Universal Interpolation Inequalities

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Contents

1. Universal Interpolation Inequalities

THEOREM 1.1. *There exist* universal *constants A, B and C such that for every nonnegative convex function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t) = 0$ iff $t = 0$, and for every function $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} \varphi\left(\frac{|u'|}{A}\right) \leq B \int_{\mathbb{R}} \varphi(|u|) + C \int_{\mathbb{R}} \varphi(|u''|)\,. \tag{1.1}$$

PROOF. Let us consider an interval $[0, L]$ and two positive constants λ, μ with $\lambda + \mu < 1$. Let $\theta \in [0, \lambda L]$ and $\eta \in [L - \mu L, L]$ then there exists a value $\xi \in [0, L]$ such that

$$u'(\xi) = \frac{u(\theta) - u(\eta)}{\theta - \eta}$$

hence,

$$u'(x) = \frac{u(\theta) - u(\eta)}{\theta - \eta} + \int_{\xi}^{x} u'' \quad \forall x \in [0, L]$$

and

$$|u'(x)| \leq \frac{|u(\theta) - u(\eta)|}{|\theta - \eta|} + \int_0^L |u''| \quad \forall x \in [0, L]$$

which implies that

$$|u'(x)| \leq \frac{|u(\theta)|}{L(1 - \lambda - \mu)} + \frac{|u(\eta)|}{L(1 - \lambda - \mu)} + \int_0^L |u''| \qquad \forall x \in [0, L]\,.$$

Now we integrate in $\theta \in [0, \lambda L]$ and $\eta \in [L - \mu L, L]$, obtaining

$$|u'(x)| \leq \frac{1}{\lambda(1 - \lambda - \mu)L^2} \int_0^{\lambda L} |u| + \frac{1}{\mu(1 - \lambda - \mu)L^2} \int_{L - \mu L}^{L} |u| + \int_0^L |u''|\,,$$

after dividing by $\lambda\mu L^2$, for every $x \in [0, L]$.
Dividing both sides for

$$\sigma = \frac{\mu\lambda(1 - \lambda - \mu)L^2 + \lambda + \mu}{\mu\lambda(1 - \lambda - \mu)L^2}$$

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and applying φ we get

$$\begin{aligned}\varphi\left(\frac{|u'(x)|}{\sigma}\right) \leq \varphi\Bigg(&\frac{\mu}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_0^{\lambda L}|u|\\ +&\frac{\lambda}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_{L-\mu L}^{L}|u|\\ +&\frac{\mu\lambda(1-\lambda-\mu)L^2}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_0^{L}|u''|\Bigg)\end{aligned}$$

hence, using two times the Jensen's Inequality,

$$\begin{aligned}\varphi\left(\frac{|u'(x)|}{\sigma}\right) \leq\, &\frac{\mu}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\lambda L}\int_0^{\lambda L}\varphi\left(|u|\lambda L\right)\\ +&\frac{\lambda}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\mu L}\int_{L-\mu L}^{L}\varphi\left(|u|\mu L\right)\\ +&\frac{\mu\lambda(1-\lambda-\mu)L}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_0^{L}\varphi\left(|u''|L\right)\end{aligned}$$

for every $x \in [0, L]$.
Finally we integrate in $x \in [0, L]$

$$\begin{aligned}\int_0^L\varphi\left(\frac{|u'|}{\sigma}\right) \leq\, &\frac{\mu}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\lambda}\int_0^{\lambda L}\varphi\left(|u|\lambda L\right)\\ +&\frac{\lambda}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\mu}\int_{L-\mu L}^{L}\varphi\left(|u|\mu L\right)\\ +&\frac{\mu\lambda(1-\lambda-\mu)L^2}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_0^{L}\varphi\left(|u''|L\right)\end{aligned}$$

and adding on disjoint intervals of length L whose union gives $\mathbb{R}$, we get

$$\begin{aligned}\int_{\mathbb{R}}\varphi\left(\frac{|u'|}{\sigma}\right) \leq\, &\frac{\mu}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\lambda}\int_{\mathbb{R}}\varphi\left(|u|\lambda L\right)\\ +&\frac{\lambda}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\frac{1}{\mu}\int_{\mathbb{R}}\varphi\left(|u|\mu L\right)\\ +&\frac{\mu\lambda(1-\lambda-\mu)L^2}{\mu\lambda(1-\lambda-\mu)L^2+\lambda+\mu}\int_{\mathbb{R}}\varphi\left(|u''|L\right).\end{aligned}$$

In the special case when $\lambda = \mu < 1/2$ we have

$$\int_{\mathbb{R}}\varphi\left(\frac{|u'|}{\sigma}\right) \leq \frac{2}{\lambda^2(1-2\lambda)L^2+2\lambda}\int_{\mathbb{R}}\varphi\left(|u|\lambda L\right) + \frac{\lambda(1-2\lambda)L^2}{\lambda(1-2\lambda)L^2+2}\int_{\mathbb{R}}\varphi\left(|u''|L\right)$$

where σ is given by

$$\sigma = \frac{\lambda(1-2\lambda)L^2+2}{\lambda(1-2\lambda)L^2}.$$

Choosing $L = L(\lambda)$ in order that $\lambda(1-2\lambda)L^2 = 2$ we get $\sigma = 2$ and

$$\int_{\mathbb{R}}\varphi\left(\frac{|u'|}{2}\right) \leq \frac{1}{2\lambda}\int_{\mathbb{R}}\varphi\left(|u|\lambda L\right) + \frac{1}{2}\int_{\mathbb{R}}\varphi\left(|u''|L\right)$$

which, by multiplication of u by $L/2$, becomes

$$\int_{\mathbb{R}}\varphi\left(\frac{|u'|L}{4}\right) \leq \frac{1}{2\lambda}\int_{\mathbb{R}}\varphi\left(\frac{|u|\lambda L^2}{2}\right) + \frac{1}{2}\int_{\mathbb{R}}\varphi\left(\frac{|u''|L^2}{2}\right)$$

and rescaling $u(x)$ to $u(tx)$,

$$\int_{\mathbb{R}} \varphi\left(\frac{|u'|tL}{4}\right) \leq \frac{1}{2\lambda}\int_{\mathbb{R}} \varphi\left(\frac{|u|\lambda L^2}{2}\right) + \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{|u''|t^2 L^2}{2}\right).$$

Then if $t = \sqrt{\lambda}$

$$\int_{\mathbb{R}} \varphi\left(\frac{|u'|\sqrt{\lambda}L}{4}\right) \leq \frac{1}{2\lambda}\int_{\mathbb{R}} \varphi\left(\frac{|u|\lambda L^2}{2}\right) + \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\frac{|u''|\lambda L^2}{2}\right)$$

and finally dividing u by $\lambda L^2/2$ we have

$$\int_{\mathbb{R}} \varphi\left(\frac{|u'|}{2\sqrt{\lambda}L}\right) \leq \frac{1}{2\lambda}\int_{\mathbb{R}} \varphi(|u|) + \frac{1}{2}\int_{\mathbb{R}} \varphi(|u''|).$$

With some computations and taking into account that $\lambda(1-2\lambda)L^2 = 2$, we conclude that for every $A > 2\sqrt{2}$ the following inequality holds

$$\int_{\mathbb{R}} \varphi\left(\frac{|u'|}{A}\right) \leq \frac{A^2}{A^2-8}\int_{\mathbb{R}} \varphi(|u|) + \frac{1}{2}\int_{\mathbb{R}} \varphi(|u''|)$$

which gives the thesis.

Notice that if $A \searrow 2\sqrt{2}$ then $B = \frac{A^2}{A^2-8} \to +\infty$. Moreover, we have an explicit $C = 1/2$. □

REMARK 1.2. The line of the proof is taken by Adams [**1**], Lemma 2.10.

REMARK 1.3. This kind of inequalities between derivatives are also known as *Gagliardo–Nirenberg Inequalities*, see [**2, 3**].

REMARK 1.4. The universal constants we found are not necessarily the best ones, in particular we do not know what is the *best* A or the infimum of the admissible ones. The proof shows that

$$\inf\{A \in \mathbb{R} \,|\, \text{Theorem 1.1 holds with constants } A, B, C\} \leq 2\sqrt{2}.$$

In particular we would be interested in a counterexample showing that the constant A cannot be taken to be 1.

References

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