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On the structure of complete manifolds of nonnegative curvature

By Jeff Cheeger and Detlef Gromoll*

A central problem in riemannian geometry is the study of complete manifolds $M$ whose sectional curvature $K$ is of a fixed sign. Until recently, when $K \geq 0$, attention has, for the most part, centered on the case where $M$ is actually compact. An exception, however, is the work of Cohn-Vossen (influenced to some extent by ideas of H. Hopf), which deals with the complete case in dimension 2 (see [8] for references). Cohn-Vossen obtained the following classification.

Theorem. In dimension 2, a noncompact complete manifold of nonnegative curvature is either diffeomorphic to $\mathbb{R}^2$ or is flat.

The proof of this result was based on delicate arguments involving the Gauss-Bonnet Theorem for geodesic polygons. Such arguments do not easily generalize to higher dimensions. However, there is another important feature of Cohn-Vossen's theory which does admit generalization—the notion of what we now call a simple point in a riemannian manifold. A point $p \in M$ is said to be simple if there are no geodesic loops in $M$ closed at $p$. Cohn-Vossen showed that the set of simple points is always nonempty for $K > 0$, $\dim M = 2$. From Morse Theory it is easy to see that the existence of a simple point implies that $M$ is contractible. However, since Morse Theory was not available to Cohn-Vossen, simple points were probably of geometrical, rather than topological interest to him.

More recently, Gromoll and Meyer [8] were able to show that in all dimensions, simple points exist provided $K > 0$. This approach begins with the construction of compact totally convex sets. A subset $C \subset M$ is said to be totally convex if any geodesic segment whose end points lie in $C$ is contained in $C$. The arguments of [8] are based on second variation techniques and do not as they stand carry over to the case $K \geq 0$. Our starting point here, is the observation that by use of more global arguments involving the generalized version of Toponogov's theorem on geodesic triangles, the constructions of [8] may be shown to yield totally convex sets in case only $K \geq 0$.

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The strong interaction of nonnegative curvature and convexity in the large turns out to be of basic importance in various applications. A main result is to produce the natural generalization of a simple point—a compact totally geodesic and totally convex submanifold $S$ which we call the soul of $M$. Again from Morse Theory, it is not difficult to see that the inclusion $S \subseteq M$ is a homotopy equivalence. In fact we prove the stronger theorem that $M$ is diffeomorphic to the normal bundle of $S$. To a great extent this has the result of reducing the complete case to the compact case. In addition, there are various other applications of our techniques, one of which is essentially to reduce the compact case to the compact simply connected case. Many of these results were announced in [4].

The body of the paper is divided into ten sections as follows:

1. Basic constructions,
2. Topological equivalence $\nu(S) \to M$,
3. Vanishing of mixed curvatures along $S$,
4. Case of codim 1 and Toponogov’s Splitting Theorem,
5. Global behavior of geodesics,
6. Groups of isometries,
7. Locally homogeneous spaces,
8. Classification in dimension $\leq 3$,
9. Coverings and the fundamental group,
10. Some examples and problems.

We refer to [3] and [7] for the tools in riemannian geometry that will be used. (Compare also the preceding article [8].) In the following, $M$ always denotes a complete riemannian manifold with curvature $K_\sigma \geq 0$ for all tangent plane sections $\sigma$. $M$ will be noncompact except in Section 9. $\rho(p, q)$ is the metric distance between points $p, q \in M$, $\exp_p : M_p \to M$ the exponential map on the tangent space $M_p$ of $M$ at $p$ and $L(c)$ the arc length of a sectionally smooth curve $c : [\alpha, \beta] \to M$.

1. Basic constructions

The main purpose of this section is to construct the totally geodesic submanifold $S$ of $M$ described in the introduction. Since the most important notion we use is that of a totally convex set (t.c.s.), we repeat the definition for the sake of completeness. A nonempty subset $C$ of $M$ will be called totally convex if for any $p$, $q \in C$ and any geodesic $c : [0, 1] \to M$ from $p$ to $q$, we have $c[0, 1] \subseteq C$. One reason for the significance of total convexity lies in the fact that topologically, a t.c.s. is very similar to the ambient space. Note
that a point is not totally convex in general and for many riemannian manifolds, proper t.c.s. do not exist at all.

We begin by recalling the theorem of Toponogov (for a proof see [1], [3], [7]).

**Theorem 1.1 (Toponogov).** Let $c_i$ be the sides of a geodesic triangle in the complete riemannian manifold $M$ and let $\alpha_i$ denote the angle at the vertex opposite $c_i$, $i = 0, 1, 2$. Assume that $c_1, c_2$ are minimal geodesics and suppose $L(c_i) + L(c_2) \geq L(c_0)$. Let the sectional curvature satisfy $K \geq H$ everywhere and if $H > 0$ assume further that $L(c_i) \leq \pi/\sqrt{H}$. Then there exists a geodesic triangle in the canonical simply connected 2-dimensional space $M_H$ of constant curvature $= H$, with sides $\overline{c}_i$ and angles $\overline{\alpha}_i$, such that $L(c_i) = L(\overline{c}_i)$ for all $i$, and $\alpha_i \geq \overline{\alpha}_i$ for $i = 1, 2$.

We will say that a geodesic is normal if it is parameterized with respect to arc length.

Recall that the normal geodesic $c : [0, \infty) \to M$ is called a ray if any segment of $c$ is minimal between its end points.

Let $c$ be a ray, $c(0) = p$. Define the open half-space

$$B_c = \bigcup_{t > 0} B_t(c(t))$$

with respect to $c$, where $B_t(c(t))$ is the open metric ball of radius $t$ centered at $c(t)$.

**Theorem 1.2 (Basic construction).** If $M$ has nonnegative curvature, then the closed complement $M - B_c$ of any half-space $B_c$ is totally convex.

**Proof.** Suppose there exists a geodesic $c_0 : [0, 1] \to M$ with end points $c_0(0), c_0(1) \in M - B_c$, but $c_0(s) \in B_c$ for some $s \in (0, 1)$. It follows from the triangle inequality that $t_s \geq t_i > 0$ implies $B_{t_s}(c(t_s)) \subseteq B_{t_i}(c(t_i))$ and hence that for $q = c_0(s) \in B_c$ there exists $t_0 > 0$ such that $q \in B_t(c(t)), t \geq t_0$. In fact setting $t_0 - \varepsilon = \rho(q, c(t_0)), \varepsilon > 0$, we have $\rho(q, c(t)) \leq \rho(q, c(t_0)) + \rho(c(t_0), c(t)) = t_0 - \varepsilon + t - t_0 = t - \varepsilon$ for all $t \geq t_0$.

Let $c_0(s)$ be a point on $c_0$ which is closest to $c(t)$. Further, consider the restriction $c'_0 = c_0|_{[0, s]},$ and minimal geodesics $c'_i$ from $c_0(s)$ to $c(t)$, $c'_i$ from $c(t)$ to $c_0(0)$. It follows from the above that for all $t \geq t_0$, $L(c'_0) \leq L(c'_0) + \varepsilon$. On the other hand, since $L(c'_0) < L(c_0)$ is finite, we will have $L(c'_0) + L(c'_0) > L(c'_0)$, for $t$ sufficiently large. Hence, we may construct a triangle in euclidean 2-space as in Theorem 1.1. In particular, for this triangle we have $\overline{\alpha}_i \leq \alpha'_i$.

Using the law of cosines in the euclidean plane, we obtain
\[ \cos \alpha^i_s = \frac{L^i(c'_i) + L^i(c'_t) - L^i(c'_t)}{2L(c'_i)L(c'_t)} \]

\[ = \frac{L(c'_i) + L(c'_t)}{2L(c'_i)} - \frac{L(c'_t) - L(c'_t)}{2L(c'_i)} + \frac{L(c'_i)}{2L(c'_t)}. \]

Now \( L(c'_i) - L(c'_t) \geq \varepsilon, \) \( L(c'_t) < L(c'_0), \) \( L(c'_t) \geq t, \) and for sufficiently large \( t \) we have \( \cos \alpha^i_s < 0, \) implying \( \pi/2 < \alpha^i_s \leq \alpha^i_t. \) However, \( c_s(s) \) is closest to \( c(t) \) and \( s \in (0, 1), \) so \( \alpha^i_t = \pi/2, \) contradiction.

**Proposition 1.3.** With \( M \) as above and \( p \in M \) there exists a family of compact totally convex sets \( C_t, \) \( t \geq 0, \) such that

1. \( t_2 \geq t_1 \) implies \( C_{t_2} \supseteq C_{t_1} \) and

\[ C_{t_1} = \{ q \in C_{t_2} \mid \rho(q, \partial C_{t_1}) \geq t_2 - t_1 \}; \]

in particular, \( \partial C_{t_1} = \{ q \in C_{t_2} \mid \rho(q, \partial C_{t_2}) = t_2 - t_1 \}. \)

2. \( \bigcup_{t \geq 0} C_t = M. \)

3. \( p \in \partial C_0. \)

Here our notation is \( \partial \tilde{A} = \tilde{A} \cap M - \tilde{A} = \tilde{A} - \tilde{A}, \) where \( \tilde{A} \) denotes the closure and \( \tilde{A} \) the interior of a subset \( A \) in \( M. \)

**Proof.** Set \( C_t = \bigcap_r (M - B_{c_r}), \) where the intersection is taken over all rays \( c \) emanating from \( p \) and \( c: [0, \infty) \to M \) denotes the restricted ray from \( c(t) \) to \( \infty \) with \( c_s(s) = c(s + t). \) Clearly \( C_t \) is totally convex and closed. If some \( C_t \) were not compact, it would contain a sequence of points \( p_i \to \infty. \)

Let \( c_i: [0, \beta_i] \to M \) be a sequence of minimal normal geodesics from \( p \) to \( p_i. \)

Each \( c_i \) is contained in the t.c.s. \( C_t. \) Then by compactness of the unit sphere in \( M_p, \) we may choose a subsequence of \( c_i \) converging to a ray \( c: [0, \infty) \to C_t. \)

But then by definition, \( c(t') \in C_t \) for \( t' > t. \) Hence, each \( C_t \) is compact.

1. Since \( B_{\varepsilon_s}(c_{t_2}(s)) = B_{\varepsilon_s}(c(s + t_2)) \subseteq B_{s + t_2 - t_1}(c(s + t_2)) = B_{s + t_2 - t_1}(c_{t_1}(s + t_2 - t_1)) \) we obtain that \( B_{\varepsilon_{t_2}} \subseteq B_{\varepsilon_{t_1}} \) for any ray \( c \) and hence \( C_{t_2} \supseteq C_{t_1}. \)

We may assume that \( t_2 > t_1. \) Given a ray \( c \) from \( p, \) we have

\[ B_{\varepsilon_{t_1}} = \{ q \mid \rho(q, B_{\varepsilon_{t_2}}) < t_2 - t_1 \}. \]

For, if \( \rho(q, B_{\varepsilon_{t_2}}) < t_2 - t_1 \) then there is \( q' \in B_{\varepsilon_{t_2}} \) such that \( \rho(q, q') < t_2 - t_1. \)

So when \( s > 0 \) is sufficiently large, \( q' \in B_s(c_{t_2}(s)) \) and, by the triangle inequality,

\[ q \in B_{s + t_2 - t_1}(c(s + t_2)) \subseteq B_{\varepsilon_{t_1}}. \]

Conversely, if \( q \in B_{\varepsilon_{t_1}} \) then, for some \( s > 0,\)

\[ q \in B_{s + t_2 - t_1}(c_{t_1}(s + t_2 - t_1)) = B_{s + t_2 - t_1}(c_{t_2}(s)) \supseteq B_{\varepsilon_{t_2}}(c_{t_2}(s)). \]

Hence, \( \rho(q, B_{\varepsilon_{t_2}}) \leq \rho(q, B_{\varepsilon_{t_2}}) < t_2 - t_1. \)
If $A_i$ is an arbitrary collection of nonempty subsets of $M$ it follows that 
\[ \rho(q, \bigcup A_i) < t_{2} - t_{1} \] 
if and only if there exists $j$ with \[ \rho(q, A_j) < t_{2} - t_{1} \].
Therefore
\[ \bigcup_{c} B_{r_{t_{1}}} = \{ q \mid \rho(q, \bigcup_{c} B_{r_{t_{1}}}) < t_{2} - t_{1} \} , \]
where $c$ ranges over all rays emanating from $p$, and
\[ C_{t_{1}} = \{ q \mid \rho(q, \bigcup_{c} B_{r_{t_{1}}}) \geq t_{2} - t_{1} \} . \]

Note that for any $A \subset M$, $q \in M - A$, one has $\rho(q, A) = \rho(q, \partial A)$, and
\[ \partial A = \partial(M - A) . \] Let $A = \bigcup_{c} B_{r_{t_{1}}}$. This completes the proof.

(2) It follows from the triangle inequality that for any $q \in M$ we have
\[ q \in C_{t} \] whenever \[ t \geq \rho(q, p) . \]

(3) This is an immediate consequence of the construction.

Proposition 1.3 provides an expanding filtration of $M$ by compact totally convex sets. However, it is even more important to be able to construct minimal t.c.s. by means of a contraction procedure. In order to do this we need some preliminaries on the structure of convex sets in an arbitrary riemannian manifold $M$. Actually the full strength of the structure theorem, (Theorem 1.6), will not be needed.

Because all of our arguments use only local (as opposed to total) convexity, we find it convenient to make the following definition: A subset $A$ of $M$ is called strongly convex if for any $q$, $q' \in A$ there is a unique minimal geodesic segment $c: [0, 1] \to M$ from $q$ to $q'$ and $c[0, 1] \subset A$. Recall [7] that there exists a positive continuous function $r: M \to (0, \infty]$, the convexity radius, such that any open metric ball $B(p') \subset B_{r(p)}(p)$ is strongly convex. We say that a set $C \subset M$ is convex if for any $p \in \overline{C}$ there is a number $0 < \varepsilon(p) < r(p)$ such that $C \cap B_{\varepsilon(p)}(p)$ is strongly convex. A totally convex set is of course convex and connected. Notice also that the closure of a convex set is again convex.

Let $C$ be a connected nonempty convex subset of an arbitrary riemannian manifold $M$. We are going to show that $\overline{C}$ the closure of $C$, is an imbedded $k$-dimensional topological submanifold of $M$ with totally geodesic connected interior and (possibly nonsmooth) boundary, which might be empty. We begin by constructing a submanifold $N$ which will turn out to be the interior of $\overline{C}$.

Let $0 \leq k \leq n$ denote the largest integer such that the collection $\{ N_\alpha \}$ of smoothly imbedded $k$-dimensional submanifolds of $M$ which are contained in $C$ is nonempty. Let $N = \bigcup N_\alpha$. Let $p \in N$. Then $p \in N_\alpha(p)$ for some $\alpha(p)$ and since $N_{\alpha(p)}$ is a smoothly imbedded submanifold, there is a neighborhood $U \subset N_{\alpha(p)} \cap B_{1/2\varepsilon(p)}(p)$ of $p$ in $N_{\alpha(p)}$ and a positive $\delta < 1/2 \cdot \varepsilon(p)$ such that the
exponential map restricted to the set of vectors normal to \( U \) and of length \(< \delta\) is a diffeomorphism onto a neighborhood \( T_\delta \) of \( p \) in \( M \). In order to show that \( N \) is an imbedded submanifold, it will suffice to show that \( N \cap T_\delta = U \). In fact, if \( q \in (C \cap T_\delta) - U \), and \( q' \) is a point of \( U \) closest to \( q \), then the minimal geodesic from \( q \) to \( q' \) is perpendicular to \( U \). Then for a sufficiently small open neighborhood \( q' \in U' \subset U \), the minimal geodesic from \( q \) to any \( q'' \in U' \) intersects \( U' \) transversally. It follows that the cone \( \{ \exp (tu) \mid u \in M_q, \| u \| < \varepsilon(q), \exp (u) \in U', 0 < t < 1 \} \) is a \((k+1)\)-dimensional smooth submanifold of \( M \) which is contained in \( C \) by convexity. But this contradicts the definition of \( k \). By convexity of \( C \) and the existence of \( T_\delta \), it is immediate that the submanifold \( N \) is totally geodesic.

The following lemma contains most of the technical information that will be needed.

**Lemma 1.4.** Let \( C \) be convex and connected, \( p \in C \cap N \), \( p' \in B_{1/4 \varepsilon(p)}(p) \cap C \), and \( q \in B_{1/4 \varepsilon(p)}(p) \cap N \). Let \( c \) be the normal geodesic in \( M \) such that \( c(0, \varepsilon) \) is the minimal segment from \( q \) to \( p' \), where \( \varepsilon = \rho(q, p') \). Then, \( c(0, \varepsilon) \subset N \) and hence \( p' \in N \). If furthermore \( p' \in N \), then \( c(\varepsilon) \subset C \) for all \( \varepsilon < s < \varepsilon + 1/4 \cdot \varepsilon(p) \).

**Proof.** Let \( W \) be a sufficiently small \((k-1)\)-dimensional hypersurface of \( B_{1/4 \varepsilon(p)}(p) \cap N \) through \( q \) which is transversal to \( c \). Let \( 0 < \varepsilon < \varepsilon + 1/4 \cdot \varepsilon(p) \) and \( \bar{p} = c(\varepsilon) \in C \). The cone \( V = \{ \exp (tw) \mid w \in M_{\bar{p}}, \| w \| < \varepsilon(p), \exp (w) \in W, 0 < t < 1 \} \) is a smooth \( k \)-dimensional submanifold of \( M \) and \( V \subset C \), since \( C \) is convex. Hence, \( V \subset N \) by construction of \( N \). In particular, taking \( \bar{p} = p' \) we have \( c(0, \varepsilon) \subset N \). On the other hand, \( \varepsilon < \varepsilon < 1/4 \cdot \varepsilon(p) \) would imply \( p' \in N \), which is contrary to what we have assumed.

**Lemma 1.5.** \( N \) is connected and \( C \subset N \).

**Proof.** Let \( N_0 \) be a connected component of \( N \). It follows that \( C \subset N_0 \). Otherwise, since \( C \) is connected, there exists \( p \in C \cap N_0 \), \( p' \in B_{1/4 \varepsilon(p)}(p) \cap (C - N_0) \) and \( q \in B_{1/4 \varepsilon(p)}(p) \cap N_0 \). But \( p' \in N_0 \) by Lemma 1.4. Suppose, \( N_1 \) is another connected component of \( N \). Since both \( N_0 \) and \( N_1 \) are dense in \( C \), we must have \( N_0 = N_1 = N \). This completes the proof.

It is quite easy to show that \( C \) and \( \bar{C} \) have the same submanifold \( N \) of interior points. We now give an indication of the proof of the structure theorem. The details are quite straightforward.

**Theorem 1.6.** Let \( C \) be a closed connected convex subset of an arbitrary riemannian manifold. Then \( C \) carries the structure of an imbedded \( k \)-dimensional submanifold of \( M \) with smooth totally geodesic interior \( \text{int} \ C = N \) and (possibly nonsmooth) boundary \( \partial C = \bar{N} - N \).
Proof. By Lemma 1.5, we have \( C = \bar{N} \). Let \( p \in \bar{N} - N \) and \( q \in B_{1/5\varepsilon(p)}(p) \cap N \). Set
\[
U = \{ q \in N | ||v|| = 1 \text{ and } \exp_q sv \in \bar{N} - N \text{ for some } 0 < s < \varepsilon(p)/5 \}.
\]
Using Lemma 1.4, it follows that for fixed \( v \in U \), the value of \( s \) is unique. We denote it by \( f(v) \). Again, by use of Lemma 1.4, it is not hard to show that \( U \) is open in \( S^{n-1} \subset N_q \) and that \( f(v) \) is continuous. Then the map \( \Phi: (0, 1] \times U \to C \) defined by \( \Phi(t, v) = \exp_{t \cdot f(v)} \cdot v \) is a homeomorphism onto the neighborhood of \( p \) in \( C \).

q.e.d.

For \( p \in \partial C \), we will have need of an object \( C_p \), which generalizes the notion of the subspace \( N_q \) tangent to \( N \) at \( q \in N \). The tangent cone \( C_p \) at \( p \in C \) is by definition the set
\[
\{ v \in M_p | \exp_p \cdot v/||v|| \in N \text{ for some positive } t < r(p) \} \cup \{0\}.
\]

Let \( \hat{C}_p \) denote the subspace of \( M_p \) generated by \( C_p \). Clearly if \( p \in N \) then \( C_p = \hat{C}_p = N_q \). If, on the other hand, \( p \in \partial C \), then by Lemma 1.5 there exists \( q \in B_{1/4\varepsilon(p)}(p) \cap N \). Letting \( c \) denote the minimal normal geodesic from \( p \) to \( q \), by Lemma 1.4 it follows that \( \hat{c}(0) \in C_p \) but \(-\hat{c}(0) \notin C_p \). Hence, in this case \( C_p \neq \hat{C}_p \). Moreover, if \( q' \in B_{1/4\varepsilon(p)}(p) \cap N \) and for sufficiently small \( t \), \( \gamma_t \) denotes the minimal normal geodesic from \( c(t) \) to \( q' \), then \( \gamma_t(0) = \lim_{t \to 0} \gamma_t(0) \). Since \( N \) is totally geodesic, \( N_{e(t)} \) is invariant under parallel translation along \( c \). Letting \( P_{[1]} \) denote parallel translation along \( [ ] \), we have \( \hat{C}_p \subset P_{-e}(N_q) \). However, as in the proof of Lemma 1.4 forming the geodesic cone at \( p \) and a small hypersurface \( W \) of \( N \) through \( q \) and transversal to \( c \), we see that \( C_p - 0 \) is open in \( P_{-e}(N_q) \). Therefore, \( \hat{C}_p = P_{-e}(N_q) \). In particular \( \dim C_p = \dim \hat{C}_p = \dim N \). Moreover, if \( p' \in B_{1/4\varepsilon(p)}(p) \cap \partial C \) and \( q \in B_{1/4\varepsilon(p)}(p) \cap N \) (Lemma 1.5) then for minimal segments \( c, c' \) from \( p, p' \) to \( q \), we have \( \hat{C}_p = P_{-e\cdot e}(\hat{C}_p) \). It follows that \( \hat{C}_p \) varies continuously with \( p \).

Lemma 1.7. Let \( C \subset M \) be closed and convex. Suppose that there exist \( p \in \partial C \), \( q \in \text{int} C \), and a minimal normal geodesic \( c: [0, d] \to C \) from \( q \) to \( p \) such that \( L(c) = \rho(q, \partial C) \). Then \( C_p - 0 \) is the open half-space
\[
H = \{ v \in \hat{C}_p | \not\prec(v, -\hat{c}(d)) < \pi/2 \}.
\]

Proof. Take \( s < d \) such that \( \rho(c(s), p) < 1/2 \cdot \varepsilon(p) \). Clearly \( c| [s, d] \) realizes the distance from \( c(s) \) to \( \partial C \) and hence \( B_{d-s}(c(s)) \cap \partial C = p \). It follows easily that \( C_p \supset H \). Conversely, let \( q' \in B_{\varepsilon(p)}(p) \cap C \) and let \( \hat{c} \) denote the minimal geodesic from \( p \) to \( q' \). If \( \not\prec(\hat{c}(0), -\hat{c}(d)) > \pi/2 \), then \( -\hat{c}(0) \) points into \( B_{d-s}(c(s)) \). Hence, by Lemma 1.4 we would have \( p \in \text{int} C \). Since, as we have previously remarked, \( C_p - 0 \) is open in \( \hat{C}_p \), this suffices to complete the proof.

q.e.d.
Any half-space \( H \subseteq \hat{C}_p \) with the property \( C_p \subset H \) will be called a supporting half-space for \( C \) at \( p \in \partial C \). If \( p \in \partial C \) and \( q \in \text{int} \ C \) with \( \rho(p, q) < \varepsilon \) (Lemma 1.5), then the closest point \( p' \in \partial C \) to \( q \) satisfies \( \rho(p, p') < 2\varepsilon \). Hence by Lemma 1.7, the set of points having a unique supporting half-space is dense in \( \partial C \). Moreover, if \( p \in \partial C \) is arbitrary, and \( p_i \to p \) where \( p_i \in \partial C \) has a supporting half-space \( H_i \), then by local compactness of the unit sphere bundle, we may assume \( H_i \) converges to some half-space \( H \). By an argument similar to the discussion following Theorem 1.6, it follows that \( H \) is a supporting half-space for \( C \) at \( p \). The following proposition, which is not used elsewhere in this paper, describes the situation more precisely.

**Proposition 1.8.** \( C_p - 0 = \bigcap H_j \) where the intersection is taken over all supporting half-spaces at \( p \in \partial C \). In particular, \( C_p \) is a convex cone.

**Proof.** By definition \( C_p - 0 \subseteq \bigcap H_j \). Given \( v \in \hat{C}_p - C_p \), it suffices to find a supporting half-space \( H \) with \( v \in H \). As we have remarked above, if \( p_i \to p \) then \( \hat{C}_{p_i} \to \hat{C}_p \) and if \( H_i \) are supporting half-spaces such that \( H_i \to H \) then \( H \) is a supporting half-space at \( p \). Suppose first that for \( t > 0 \), \( \exp_{p_i}tv \in C \). Let \( \exp_{p_i}tv \to v \). We may assume that \( v_i \to v \) for all \( i \). Let \( t_i \) be the maximum value of \( t \) for which \( \exp_{p_i}tv \subseteq [0, t]v_i \subseteq C \). Then since \( C \) is closed, \( t_i \to 0 \). In this case set \( p_i = \exp_{p_i}tv_i \) and \( v_i = \frac{d}{dt} \exp_{p_i}tv_i |_{t_i} \). Then \( -v_i \in C_{p_i} \), so there exists a half-space \( H_i \) with \( v_i \in H_i \). Therefore, we may assume that for some \( \varepsilon > 0 \), \( \exp_{p_i}tv \in C \) for \( t \leq \varepsilon \). Then since \( v \in C_p \), \( \exp_{p_i}tv \in \partial C \) for \( t \leq \varepsilon \). However, then there is a sequence \( \hat{C}_p \ni v_j \to v \) with \( \exp_{p_i}tv_j \in C \) for \( t > 0 \). Otherwise there would be a neighborhood \( U \ni \varepsilon/2 \cdot v \), open in \( \hat{C}_p \) such that \( \exp_p U \subseteq C \). But then \( \exp_{p_i}tv_j = \hat{C}_{p_i}tv_j \) contradicting \( \exp_p \varepsilon/2 \cdot v \in \partial C \). Let \( H = \lim H_j \) with \( v_j \in H_j \). Then \( v \in H \).

The following theorem is the key to constructing minimal t.c.s. For \( C \) closed and convex set with \( \partial C \neq \emptyset \), we set

\[
C^a = \{ p \in C \mid \rho(p, \partial C) \geq a \},
\]

\[
C^{\max} = \bigcap_{C^a \neq \emptyset} C^a.
\]

**Theorem 1.9.** Let \( M \) have nonnegative curvature and let \( C \) be closed and convex, resp. totally convex, \( \partial C \neq \emptyset \). Then

1. for any \( a \), \( C^a \) is convex, resp. totally convex,
2. \( \dim C^{\max} < \dim C \).

Theorem 1.9 is actually a corollary of the following more general theorem.

**Theorem 1.10.** With the hypothesis of Theorem 1.9 let \( \psi : C \to R \) be defined by \( \psi(x) = \rho(x, \partial C) \). Then for any normal geodesic segment \( c \) contained
in $C$, the function $\psi \circ c(t)$ is (weakly) convex, i.e.

$$\psi \circ c(at_1 + \beta t_2) \geq \alpha \cdot \psi \circ c(t_1) + \beta \cdot \psi \circ c(t_2),$$

where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Moreover, suppose $\psi \circ c(t) = d$ is constant on some interval $[a, b]$. Let $V(s)$ denote the parallel field along $c$ $[a, b]$ such that $V(a) = c_s(0)$ where $c_s$ is any minimal geodesic from $c(a)$ to $\partial C$. Then for any $s$, $\exp_{c(s)} t V(s) | [0, d]$ is a minimal geodesic from $c(s)$ to $\partial C$ and the rectangle $\varphi : [a, b] \times [0, d] \to M$ defined by $\varphi(s,t) = \exp_{c(s)} t \cdot V(s)$ is flat and totally geodesic.

Theorem 1.10 implies Theorem 1.9 as follows: If $c : [0, d] \to C$ were a normal geodesic segment such that $c(0), c(d) \in C^s$ but $c[0, d] \not\subset C^s$ then $\psi \circ c$ would have a strict interior minimum which is impossible for a convex function. This proves (1), and (2) is then trivial.

Proof of Theorem 1.10. For simplicity we will assume that $C$ is totally convex. If $C$ is convex the proof must be slightly modified and we leave this to the reader. The second part of the theorem follows easily from (a) below.

Let $c : [a, b] \to C$ be a normal geodesic. For $a < s < b$ let $c_s : [0, d] \to C$ be a minimal connection from $c(s)$ to $\partial C$. It will suffice to show that on some interval $(s - \delta, s + \delta)$, $\psi \circ c$ is bounded above by the linear function $d - \cos \alpha(s - s)$ where $\alpha = \langle \dot{c}(s), \dot{c}_s(0) \rangle$. It is only necessary to consider $s > s$. Following an idea suggested by Berger’s proof of Toponogov’s Theorem we distinguish three cases.

(a) $\alpha = \pi/2$. Let $E(t)$ denote the parallel field along $c_s$ generated by $\dot{c}(s)$. By a version of Rauch’s comparison theorem (see [1] or [3], [14], Rauch II), it follows that there exists $\delta > 0$ such that for $s \leq s \leq s + \delta$, the curve $c_s(t) = \exp_{c_s(s)} (s - s) \cdot E(t)$ has length $\leq d$ with equality holding for some $s'$ if and only if $c_s(t), [s, s'] \times [0, d] \to C$, defines a flat totally geodesic rectangle. Clearly, (a) follows unless for some fixed $s$, the curve $c_s(t) \subset \text{int} C$ for $0 \leq t \leq d$. On the other hand, by Lemma 1.7, this is impossible for $t = d$.

(b) $\alpha > \pi/2$. In this case let $E(0)$ be the convex combination of $\dot{c}_s(0)$ and $\dot{c}(s)$ which is perpendicular to $\dot{c}_s(0)$. Constructing a family $c_s(t)$ as above it follows as in (a) for small $\delta$ and $s \leq s \leq s + \delta$, that

$$\rho(\exp_{c_s(s)} (\cos(\alpha - \pi/2)(s - s) \cdot E(0), \partial C) \leq d.$$

But by the ordinary Rauch comparison theorem,

$$\rho(c(s), \exp_{c(s)} \cos(\alpha - \pi/2)(s - s) E(0)) \leq (s - s) \sin(\alpha - \pi/2) = (s - s) \cos \alpha.$$

So by the triangle inequality, $\rho(c(s), \partial C) \leq d - (s - s) \cos \alpha$. 

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(c) $\alpha < \pi/2$. Let $a_s$ be the minimal segment from $c_s$ to $c(s)$ where $a_s(0) = c_s(t_s)$. Then $\dot{a}_s(0) \perp \dot{c}_s(t_s)$. As above, $\rho(c(s), \partial C) \leq d - t_s$. By Rauch's comparison theorem and the law of cosines

$$\rho^2(c_s(t_s), c(s)) \leq t_s^2 + (s - s)^2 - 2t_s(s - s) \cos \alpha.$$  

Since the angle at $c_s(t_s)$ is a right angle, we also have $(s - s)^2 \leq \rho^2(c_s(t_s), c(s)) + t_s^2$. Adding these two inequalities yields $2t_s(s - s) \cos \alpha \leq 2t_s^2$ or $(s - s) \cos \alpha \leq t_s$. Thus again $\rho(c(s), \partial C) \leq d - \cos \alpha \cdot (s - s)$.

Now choose a compact t.c.s. $C$ as in Proposition 1.3 such that $\partial C \neq \emptyset$. By iterating Theorem 1.9, we may construct a flag of compact t.c.s., where each set consists of all points at maximal distance from the boundary of the preceding set and is of lower dimension. Thus we have by induction:

**Theorem 1.11.** $M$ contains a compact totally geodesic submanifold $S$ without boundary which is totally convex, $0 \leq \dim S < \dim M$.

In particular, $S$ has nonnegative curvature.

A manifold $S$, constructed as above, will be called a soul of $M$. The soul may be a point, and in fact this is always the case if $K > 0$. In the terminology of [8] such a soul is called a simple point. We remark that any compact $S$ of nonnegative curvature arises as soul of some $M$. One may just take $M$ to be the riemannian product of $S$ and euclidean space $R^k$, $k > 0$. It is not very hard to give examples of manifolds $M$, where codim $S$ steps are actually necessary to find a soul $S$ by the above iteration procedure. The construction of $S$ depended on the choice of some arbitrary $p \in M$. A different choice of $p$ will sometimes lead to a different soul. Still, when there is no danger of confusion we will speak of the soul of $M$ and denote it by $S$. In general not all compact totally convex compact submanifolds without boundary of $M$ can be obtained from the basic construction. The paraboloid $z = x^2 + y^2$ in $R^3$ furnishes an example of this phenomenon; compare [8].

It will be convenient to change our notation slightly by reindexing the expanding and contracting families of t.c.s. constructed in this section. Let $C_0$ be the t.c.s. of Proposition 1.3. In case $\dim C_0 = \dim M$ let us now agree to denote $C_0^{\max} = C_0^0$ by $C_0$. Then what was previously denoted by $C_i$ becomes $C_{t+\alpha_0}$ and $C_i^0$ becomes $C_{\alpha_0}$. Conclusions (1), (2) of Proposition 1.3 hold for the family as reindexed. In addition, with our new notation $\dim C_i = \dim M$, $t > 0$, and $\dim C_0 < \dim M$. Moreover, we have the flag of t.c.s. $C_0 = C(1) \supset \cdots \supset C(k) = S$, where $C(i + 1) = C(i)^{\max}$.
2. Topological equivalence $\nu(S) \to M$

We shall see in this section that the topology of the soul $S$ determines the topological structure of $M$.

**Theorem 2.1.** Let $C$ be a compact totally convex set in $M$, $\partial C = \emptyset$. Then the inclusion $C \subset M$ is a homotopy equivalence.

*Proof.* We use Morse Theory and consider the space $\Omega_c$ of all absolutely continuous paths $c: [0, 1] \to M$ such that $c(0), c(1) \in C$ and the energy $E(c)$ exists. Topology and Hilbert manifold structure of $\Omega_c$ are introduced as in the case of two fixed end points. It follows from the completeness of $M$ and the compactness of $C$ that $\Omega_c$ is complete and condition (C) of Palais and Smale holds for $E$. (One may also use the canonical finite dimensional approximation of $\Omega_c$ (curves of energy $\leq a$) by a space of broken geodesics.) The critical points of $E$ are precisely the geodesics in $\Omega_c$ perpendicular to $C$ at $c(0)$ and $c(1)$. Since $C$ is totally convex, the constant geodesics in $C$ are the only critical points forming a nondegenerate critical submanifold of index 0 in $\Omega_c$. Hence there exists a strong deformation retract $\Omega_c \to C$ along the gradient trajectories of $E$. Let $D^k$ denote the standard disc in euclidean space $\mathbb{R}^k$. A map $(D^k, \partial D^k) \to (M, C)$ induces a map $(D^{k-1}, \partial D^{k-1}) \to (\Omega_c, C)$ and vice versa, $k \geq 1$. So $\pi_k(M, C)$ is trivial for all $k \geq 1$.

By Theorem 2.1 all compact t.c.s. of $M$ with empty boundary are homotopy equivalent and thus in particular have the same dimension. Using Theorem 1.9 it then follows that if $K_M \geq 0$, the compact t.c.s. without boundary are precisely the sets which are minimal with respect to the property of being totally convex. We do not know in general whether all such sets are isometric or even homeomorphic. However, for souls, we have the following result.

**Theorem 2.2.** Let $S$ be a soul of $M$, i.e. $S$ arises from the basic construction. Then $M$ is diffeomorphic to the normal bundle $\nu(S)$ of $S$ in $M$.

This theorem shows that up to diffeomorphism, the complete manifolds of nonnegative curvature are vector bundles over compact manifolds of nonnegative curvature. The diffeomorphism can be chosen to induce the identity on $S$. In particular, $M$ is diffeomorphic to euclidean space when $S$ is a point; compare [8]. Although the compact case is still open to a very large extent, it seems reasonable to conclude that most open manifolds do not admit a complete metric of nonnegative curvature.

In case $M$ is simply connected and $\dim M \geq 6$, Theorem 2.2 can be proved by the open $h$-cobordism theorem. The argument makes use of the end struc-
ture of $M$ as described in Section 4. Here we give a more geometric proof that $M$ is homeomorphic to $\nu(S)$. No connectivity or dimension assumptions are needed. With more technical work the constructions can be smoothed out to yield the desired diffeomorphism. Basically, the idea is to use a continuous filtration of $M$ by compact t.c.s. $C_i \supset S$ for defining an injective crooked exponential map $\nu(S) \to M$ by means of broken geodesics. Such a map will be comparatively close to the original exponential map $\exp \mid \nu(S)$ of $S$, which is not injective in general. The procedure gets somewhat complicated by the fact that the soul $S$ may be a proper subset of $C^{\text{max}}_0$.

We start with some preliminaries.

**Lemma 2.3.** For any compact subset $D$ of $M$ there exists a number $\varepsilon_D > 0$ such that for all $p \in D$ and $0 < r \leq \varepsilon_D$:

1. The metric ball $B_r(p)$ of radius $r$ centered at $p$ is the diffeomorphic strictly convex image of an euclidean ball about $0 \in M$, under the exponential map $\exp_p$.

2. If $c: [0, \gamma] \to B_r(p)$ is a nonconstant geodesic, $c_o: [0, 1] \to B_r(p)$ the minimal geodesic from $p$ to $c(0)$, and $\langle \dot{c}(0), \dot{c}_o(1) \rangle \geq 0$, then $s \to \rho(c(s), p)$ is strictly increasing on $[0, \gamma]$.

**Proof.** The standard argument is straightforward. The convexity radius is a continuous function on $M$ and assumes a positive minimum $\dot{\varepsilon}_D$ on $D$, compare [7]. Choose $0 < \varepsilon_D \leq \dot{\varepsilon}_D$ so small such that the index form $I$ is positive definite on the Jacobi fields $Y$ along normal geodesics $c_o: [0, r] \to M$ with $c_o(0) \in D$, $0 < r \leq \varepsilon_D$, $Y \perp \dot{c}_o$, $Y(0) = 0$.

Define for any compact t.c.s. $C$ and number $r \geq 0$ the set $\{C = \{p \in M \mid \rho(p, C) \leq r\}$. $C$ need not be convex. In case $\partial C \neq \emptyset$ we have the contracting family $C^* \geq 0 \leq a = \max \{\rho(p, \partial C) \mid p \in C\}$, and the flag

$$C = C(0) \supset C^{\max} = C_{a_0} = C(1) \supset \cdots \supset C(k) = S,$$

$C(i + 1) = C(i)^{\text{max}}$, $\partial S = \emptyset$, as in Section 1. This soul $S$ of $M$ will also be called the soul of $C$.

The following lemma contains all of the local information for proving Theorem 2.2. We suggest that the reader glances at the proofs of Theorem 2.5, (1) and (3), and the proof of Theorem 2.2 (which follows Theorem 2.5) before going through the details of the argument. In this way, he will get an idea of how Lemma 2.4 is to be applied. (3) of Lemma 2.4 and (2) of Theorem 2.5 are not used in the proof of Theorem 2.2.

**Lemma 2.4.** Let $C$ be a compact t.c.s. with nonempty boundary $\partial C$ and $0 < 3 \varepsilon = \varepsilon_C \leq a$, where $\varepsilon_C$ is a constant as in Lemma 2.3. Set $0 < a_i = \ldots$
max \{\rho(p, \partial C(i)) \mid p \in C(i)\} for 0 \leq i < k, so C(i + 1) = C(i)^a.

(1) There exists a number 0 < \delta \leq \varepsilon such that \rho(C(i)^a, p) < \varepsilon for all p \in C(i)^a whenever 0 \leq a \leq a' \leq a_s, a' - a < \delta, 0 \leq i < k.

(2) Given 0 \leq a < a' < a_s, a' - a < \delta, there is a homeomorphism \partial C^a \times [0, 1] \rightarrow C^a - \text{int } C^a with p \times 0 \rightarrow p. In particular, all topological hypersurfaces \partial C^a have a collar neighborhood in C.

(3) There is a strong deformation retract C^a \rightarrow C(1) for 0 \leq a_s - a < \delta.

(4) There is a homeomorphism C^a \rightarrow C \cap C(1) keeping C(1) fixed if 0 < r \leq a_s - a < \delta.

(5) Let 0 \leq a \leq a' \leq a_s, a' - a < r < \delta, and 0 < i < k. Then there is a homeomorphism C \cap [C(i)^a] \rightarrow C \cap [C(i)^a] keeping C(i + 1) fixed.

Proof. (1) It suffices to show that the function \((b, p) \mapsto \rho(C^b, p)\) is (uniformly) compact on the compact set \([0, a_s] \times C\). Suppose, \((b_s, p_s) \mapsto (b, p)\) is a sequence in \([0, a_s] \times C\) such that \(\rho(C^{b_s}, p_s) = \rho(q_s, p_s)\) converges, \(q_s \in C^{b_s}\). For a limit point \(q\) of \(q_s\), we have \(q \in C^b\), since \(\rho(q_s, \partial C) \geq b\) implies \(\rho(q, \partial C) \geq b\) by continuity of the distance from \(\partial C\). Hence, \(\lim \rho(C^{b_s}, p_s) = \rho(q, p) \geq \rho(C^b, p)\). On the other hand, there is \(\tilde{q} \in C^b, \rho(C^b, p) = \rho(\tilde{q}, p)\), and we find a sequence \(\tilde{q}_s \in C^{b_s}, \tilde{q}_s \rightarrow \tilde{q}\). Observe that for \(b < a_s\), \(C^b\) is the closure of its interior points in \(C\) (Lemma 1.5). Now \(\rho(C^{b_s}, p_s) \leq \rho(\tilde{q}_s, p_s)\), so \(\lim \rho(C^{b_s}, p_s) \leq \rho(\tilde{q}, p) = \rho(C^b, p)\).

(2) Choose \(a' < \hat{a} \leq a_s\) such that \(\hat{a} - a < \delta\). For \(p \in C^a\), we find a unique point \(h(p) \in C^{\hat{a}}, \rho(C^{\hat{a}}, p) = \rho(h(p), p)\). This is nontrivial only for \(p \in C^{\hat{a}}\).

Suppose there are two points \(q_0 \neq q_1\) in \(C^{\hat{a}}\), \(\rho(C^{\hat{a}}, p) = \rho(q_0, p) = \rho(q_1, p)\). The minimal geodesic \(c: [0, 1] \rightarrow C^{\hat{a}}\) from \(q_0\) to \(q_1\) is contained in the convex ball of radius \(\varepsilon\) centered at \(p\). Let \(c_s: [0, 1] \rightarrow C^a\) be the minimal geodesic from \(p\) to \(q_s\). Since \(c_0\) realizes the distance from \(p\) to \(C^{\hat{a}}\), we have that by the first variation formula \(\langle \dot{c}(0), \dot{c}(1) \rangle \geq 0\). Hence, \(s \mapsto \rho(c(s), p)\) is strictly increasing on \([0, 1]\) by 2) of Lemma 2.3, which gives a contradiction. It follows from the uniqueness of \(h(p)\) and the continuity of the distance \(p \mapsto \rho(C^{\hat{a}}, p)\) that \(h: C^a \rightarrow C^{\hat{a}}\) is a continuous retraction.

Denote by \(U_p\) the open ball of radius \(\varepsilon\) about 0 in \(M_p\), \(p \in C^a\). Then \(p \mapsto X_p = (\exp_p\vert U_p)^{-1} \circ h(p)\) is a continuous vector field on \(C^a\). Define a function \(\phi: \partial C^a \rightarrow (0, 1)\) as follows: The geodesic \(c_p: [0, 1] \rightarrow C^a, c_p(t) = \exp(tX_p)\), from \(p \in \partial C^a\) to \(h(p)\) intersects \(C^a\) at some \(t_p\) because of the continuity of the distance from \(\partial C^a\), and \(t_p\) is unique since \(C^a\) totally convex, compare Lemma 1.4. Set \(\phi(p) = t_p\). Clearly, \(\phi\) is continuous. Define the continuous map \(F: \partial C^a \times [0, 1] \rightarrow C^a - \text{int } C^a, F(p, t) = \exp(t\phi(p)X_p)\).
Since all geodesics \( c_p \) are minimal connections from \( p \in \partial C^a \) to \( C^\delta \), \( F \) is injective. To prove that \( F \) is also surjective choose \( q \in C^a - \text{int } C^a \). Let \( c_o: [0, 1] \to M \) be the geodesic with \( L(c_o) = \varepsilon \), \( c_o(1) = h(q) \), and \( q \in c_o([0, 1]) \). Now if \( q_i = c_o(t) \in C^a \) then we have \( h(q_i) = h(q) \). Otherwise, consider the minimal geodesic \( c: [0, 1] \to C^\delta \) from \( h(q) \) to \( h(q_i) \). As above it follows that \( \langle \dot{c}(0), \dot{c}_i(1) \rangle \geq 0 \), since \( c_o \) realizes the distance from \( q \) to \( C^\delta \). So \( s \to \rho(c(s), q_i) \) is strictly increasing on \([0, 1]\) and one has that \( \rho(h(q_i), q_i) > \rho(h(q), q_i) \), which is a contradiction. Therefore \( c_o \) is a minimal connection from \( q \) to \( C^\delta \). Now \( q_o \in C^a \) by choice of \( \varepsilon \), so \( c_o \) intersects \( \partial C^a \) at some point \( p \), and \( q \in c_o([0, 1]) \) is in the image of \( F \). Hence, \( F \) is a homeomorphism. The continuity of \( F^{-1} \) follows from the compactness of \( \partial C^a \times [0, 1] \).

(3) As in (2), we have the continuous retraction \( h: C^a \to C^a_o \), \( \rho(h(p), p) = \rho(C^a_o, p) \), and the continuous vector field \( p \to X_p = (\exp_p|U_p)^{-1} \cdot h(p) \) on \( C^a \). Define a deformation \( H: C^a \times [0, 1] \to C^a \) by \( H(p, t) = \exp(tX_p) \). Then \( H_0 = \text{id}|C^a \), \( H_t = h \), \( H_t|C(1) = \text{id}|C(1) \) for all \( t \in [0, 1] \).

(4) We contract \( C^a \) along the minimal connections from points on \( \partial C^a \) to \( C^a_o \). Note first that \( C \cap C(1) \subset C^a \). Let \( h \) and \( X \) be as above. Consider the continuous map \( F: \partial C^a \times [0, 1] \to C^a \) with \( F(p, t) = \exp(tX_p) \). It follows from the arguments in 2) that \( F|\partial C^a \times [0, 1] \) is a homeomorphism onto \( C^a - C^a_o \). Define \( g: \partial C^a \times [0, 1] \to \partial C^a \times [0, 1] \) by

\[
g(p, t) = (p, (1 - t)\left(1 - \frac{r}{\rho(h(p), p)}\right) + t) .
\]

Clearly, \( g \) maps \( \partial C^a \times [0, 1] \) homeomorphically onto \( F^{-1}(C \cap C(1)) \) and \( G: C^a \to C \cap C(1) \) with \( G(q) = F \circ g \circ F^{-1}(q) \) for \( q \in C^a - C^a_o \) and \( G(q) = q \) for \( q \in C^a_o \) is a homeomorphism.

(5) We proceed as before. Let \( D = C(i)^a \), \( D' = C(i)^a_o \). Note that \( D' \subset D \cap D' \subset D \cap D' \subset \text{int } C \). We have the continuous retraction \( h: C \cap D' \to D \), \( \rho(h(p), p) = \rho(D, p) \), and the continuous vector field \( p \to X_p = (\exp_p|U_p)^{-1} \cdot h(p) \) on \( C \cap D \). Let \( A = \{ p \in C | \rho(D, p) = r \} \) be the boundary of \( C \cap D \) in \( C \). Define the map \( F: A \times [0, 1] \to C \cap D \) by \( F(p, t) = \exp(tX_p) \). As above, \( F|A \times [0, 1] \) is a homeomorphism onto \( (C \cap D) \). Now for any \( p \in A \) there is exactly one \( t_p \in [0, 1] \) such that \( \rho(F(p, t_p), D') = r \). Then clearly, the function \( \varphi: A \to [0, 1] \) with \( \varphi(p) = t_p \) is continuous. To prove uniqueness of \( t_p \), we consider the minimal geodesic \( c: [0, 1] \to C \cap D \), \( c(s) = F(p, 1 - s) \) from \( D \) to \( p \). We need only show that \( s \to \rho(c(s), D') \) is strictly increasing on \([0, 1]\) since \( \rho(c(0), D') < r \), \( \rho(p, D') \geq r \). Choose a point \( q_s \in D' \) for any \( s \in [0, 1] \) such that \( \rho(c(s), q_s) = \rho(c(s), D') \). We have

\[
\rho(c(s), q_s) \leq \rho(c(s), q_s) \leq \rho(c(s), c(0)) + \rho(c(0), q_s) < r + \varepsilon \leq 2\varepsilon
\]
and \( \rho(c(0), q_s) \leq \rho(c(0), c(s)) + \rho(c(s), q_s) < 3\varepsilon \). Hence, fixing \( s_0 \in (0, 1] \) it follows from (2) of Lemma 2.3 that \( s \rightarrow \rho(c(s), q_{s_0}) \) is strictly increasing on \( [0, s_0] \).

Observe, if \( c_0 \colon [0, 1] \rightarrow D \) is the minimal geodesic from \( q_{s_0} \) to \( c(0) \) then by the first variation formula, \( \langle \dot{c}(0), \dot{c}_0(1) \rangle \geq 0 \). Therefore,

\[
\rho(c(s), D') \leq \rho(c(s), q_{s_0}) < \rho(c(s_0), q_{s_0}) = \rho(c(s_0), D')
\]

for \( 0 \leq s < s_0 \). Now \( g \colon A \times [0, 1] \rightarrow A \times [0, 1] \) with \( g(p, t) = (p, (1 - t)\varphi(p) + t) \) maps \( A \times [0, 1] \) homeomorphically onto \( F^{-1}(C \cap C') \). We may define a homeomorphism \( G \colon C \cap C' \rightarrow C \cap C' \) by \( G(q) = F \circ g \circ F^{-1}(q) \) for \( q \in (C \cap C') - D \) and \( G(q) = q \) for \( q \in D \).

Before finally proving Theorem 2.2 we give some general applications of the above lemma.

**Theorem 2.5.** Let \( C \) be a compact t.c.s. with nonempty boundary \( \partial C \).

1. If \( 0 < a < a_0 \) then there is a homeomorphism \( \partial C \times [0, 1] \rightarrow C - \text{int } C \) with \( p \times 0 \rightarrow p \) for \( p \in \partial C \). Furthermore, \( C \) is homeomorphic to \( C \) such that a neighborhood of \( C(1) \) is kept fixed.

2. \( C(1) \) is a strong deformation retract of \( C \).

3. Let \( S = C(k) \) be the soul of \( C \), \( \nu(S) \) the normal bundle of \( S \) in the tangent bundle of \( C \), and \( \nu_r(S) \) the associated disc bundle \( \{ v \in \nu(S) \mid ||v|| \leq r \} \).

Then there is a homeomorphism \( \nu_r(S) \rightarrow C \) inducing the identity on \( S \).

**Proof.** Determine \( \varepsilon \) and \( \delta \) as in Lemma 2.4.

1. Consider a subdivision \( 0 = b_0 < b_1 < \cdots < b_m = a \) of \( [0, a] \) such that \( b_{j+1} - b_j < \delta, 0 \leq j < m \). By (2) of Lemma 2.4 there are homeomorphisms \( F_j \colon \partial C \times [j/m, (j + 1)/m] \rightarrow C - \text{int } C^{j+1} \), \( F_j(p, j/m) = p \). Using induction we obtain a homeomorphism \( G_{j+1} \colon \partial C \times [0, (j + 1)/m] \rightarrow C - \text{int } C^{j+1} \), then \( G_m \) will be as required. Let \( G_i = F_0 \) and suppose, \( G_j \) has been constructed, \( 1 \leq j < m \). We may define \( G_{j+1}(p, t) = F_j(G_j(p, j/m), t) \) for \( j/m \leq t \leq (j + 1)/m \) and \( G_{j+1}(p, t) = G_j(p, t) \) otherwise. Now the second part of the assertion is an obvious consequence when applying the first conclusion also to \( C^{a'} \subset C \subset C \) for some \( a < a' < a_0 \).

2. This follows from (1) and 2.4, (3).

3. Combining (1) and 2.4, (4) we find a homeomorphism \( C \rightarrow C \cap C(1) \). A finite iteration of (5) in 2.4 yields a homeomorphism \( C \cap C(1) = C \cap C(1)^a \rightarrow C \cap C(1)^a \) which induces the identity on \( C(2) \). We proceed this way by induction over the flag \( C(\hat{i}) \) and finally obtain a homeomorphism \( C \rightarrow C \cap C(k) = C \cap S \) keeping \( S \) fixed. On the other hand, \( \nu_r(S) \) is mapped diffeomorphically onto \( C \cap S \) by the exponential map according to the choice of \( \delta \). This completes the argument.
Proof of Theorem 2.2. We have the expanding family $C_t$ of compact t.c.s. as reindexed at the end of Section 1 such that $C_0 = C_t(1) \supset \cdots \supset C_t(k) = S$ for $t > 0$. By (3) and (1) of Theorem 2.5 there are homeomorphisms $\nu_i(S) \to C_t$ and $\nu_{i+1}(S) - \nu_i(S) \to C_{i+1} - \text{int} C_t$ for all $i = 1, 2, \cdots$ which immediately determine a topological equivalence $\nu(S) \to M$ as claimed.

As a result of Theorem 2.5, the hypothesis $\partial C = \emptyset$ may be removed from Theorem 2.1. By use of a slight modification of the arguments and constructions of Sections 1 and 2, it is not hard to see that the assumption that $C$ is compact may also be removed. Moreover, one can show that the existence of a single proper noncompact t.c.s. puts some restrictions on the topology of $M$.

3. Vanishing of mixed curvatures along $S$

An analysis of the preceding results shows that there is a certain degree of rigidity involved in the riemannian structure of a complete open manifold $M$ of nonnegative curvature, in particular, when $M$ is not contractible and hence has a soul $S$ of positive dimension. For some time we conjectured that the rigidity could go as far as proposed by the following stronger version of Theorem 2.2: There exists a smooth bundle projection $M \to S$ which is locally isometrically a product. A weaker infinitesimal result would state that the normal bundle $\nu(S)$ of $S$ in $M$ is flat (with respect to its natural induced connection). We have now found counterexamples to these conjectures. They will be described in Section 10. Nevertheless we will show that the stronger version is true in some important special cases, namely when $S$ has dimension 1, codimension 1, or when $M$ is locally homogeneous. In addition, the following theorem holds in the general case.

Theorem 3.1. Let $C \subset M$ be compact totally convex with $\partial C \neq \emptyset$ and let $S$ be the soul of $C$. Then

$$R(u, v)v = R(v, u)u = 0$$

for any point $p \in S$ and all vectors $u, v \in C_p$, $u \in S_p$, $v \perp S_p$. Equivalently, all sectional curvatures vanish for planes spanned by a tangent vector of $S$ and a normal vector of $S$ in $C$.

This applies for example to a soul $S$ of $M$ arising from the basic construction. The equivalence of the two statements is obvious from the curvature identities. Note that $R(u, v)v$ is tangent to $S$ and $R(v, u)u$ is perpendicular to $S$ in $C$. A proof of Theorem 3.1 is obtained by combining the following two lemmas.
**Lemma 3.2.** $C$ compact totally convex, $\partial C \neq \emptyset$. Let $D = C^{\text{max}} = C^{\ominus_0}$ be the set of points in $C$ farthest away from the boundary. Then for any geodesic $c : [0, 1] \to \text{int } D$, the index of $c$ in $C$ is equal to the index of $c$ in $D$.

**Proof.** By the Morse index theorem, it suffices to show that if $Z$ is any Jacobi field along $c$ in $C$ with $Z(0) = 0$ and $Z(t_0) = 0$ for some $0 < t_0 < 1$, then $Z$ is tangent to $D$. Suppose, $Z$ is not tangent to $D$. The normal component $Y$ of $Z$ with respect to $D$ in $C$ is Jacobi, since int $D$ is totally geodesic. Obviously, $Y \neq 0$.

Choose a strictly convex open metric ball $B$ in $M$ centered about $c(t_0)$. Let $0 < t < t_0 < \bar{t} \leq 1$ be numbers such that neither $c(t)$ nor $c(\bar{t})$ are conjugate points of $c(0)$ along $c$ and furthermore, $c(t) \in B$, $c(\bar{t}) \in B$, and $\langle Y, Y \rangle_t = (1/2)D\langle Y, Y \rangle_t \leq 0$. There is a neighborhood $U$ of $c(0)$ in the tangent space of $C$ at $c(0)$ which is mapped by $\exp_{c(0)}$ diffeomorphically onto some open neighborhood $V$ of $c(t)$ in int $C$ such that $V \subset B$ and $V \cap \partial D = \emptyset$. We consider a smooth function $f : V \to \mathbb{R}$ defined as follows: For $q \in V$, one has the geodesic $c_q^t : [0, t] \to C$ from $c(0)$ to $q$ with $c_q^t(t) = \exp(tv_q)$, where $v_q = (\exp|U)^{-1}(q)$, and the minimal geodesic $c_q^\bar{t} : [t, \bar{t}] \to C$ from $q$ to $c(\bar{t})$. Denote by $c_q^t : [0, \bar{t}] \to C$ the broken geodesic composed of $c_q^t$ and $c_q^\bar{t}$. Now let $f(q) = E(c_q)$ be the energy of $c_q$, so

$$f(q) = \frac{1}{2} (\dot{c}_q^t(t)^2 + (t - \bar{t}) \dot{c}_q^\bar{t}(\bar{t})^2).$$

By the first variation formula, the gradient $\nabla f_q$ of $f$ at $q$ is given by

$$\nabla f_q = \dot{c}_q^t(t) - \dot{c}_q^\bar{t}(\bar{t}).$$

Therefore, $p = c(t)$ is a critical point of $f$. Note that $\nabla f$ is tangent to $D$ along $V \cap D$ since $D$ totally convex. The hessian form $H$ of $f$ at $p$ is easily obtained from the second variation formula,

$$H(v, w) = \langle v, Z_t^t(t) - Z_t^\bar{t}(\bar{t}) \rangle$$

for all $v, w \in C_p$, where $Z_t$ is the unique Jacobi field along $c|_{[0, t]}$ with $Z_t(0) = 0$, $Z_t(t) = w$, and $Z_{\bar{t}}$ is the Jacobi field along $c|_{[t, \bar{t}]}$ with $Z_{\bar{t}}(t) = w$, $Z_{\bar{t}}(\bar{t}) = 0$. It follows that $H$ is nondegenerate, since $\bar{t}$ is not a conjugate point of $c$. Therefore, $p$ is an isolated nondegenerate critical point of $f$. Notice that by choice of $t$ and $\bar{t}$ we have

$$H(u, u) < 0 \text{ for } u = Y(\bar{t}) \neq 0.$$  

Here we use the fact that Jacobi fields locally minimize the index form.

Now consider the local flow $\varphi_t$ of $\nabla f$ in some neighborhood of $p$ in $V$. As a nondegenerate critical point of $f$, $p$ is a hyperbolic fixed point of $\varphi_t$. So
locally, the stable manifold $W^s$ and the unstable manifold $W^u$ of $\varphi$, through $p$ exist, compare [11]. Recall that roughly speaking, $W^s$ is the set of all points $q$ for which $\lim_{t \to -\infty} \varphi_t(q) = p$, and $W^u$ is the set of all $q$ for which $\lim_{t \to +\infty} \varphi_t(q) = p$. The tangent spaces of $W^s$ and $W^u$ at $p$ are precisely the eigenspaces $C^-_p$ and $C^+_p$ in the orthogonal direct sum decomposition $C_p = C^+_p \oplus C^-_p$ which corresponds to the positive and negative part of the spectrum of the selfadjoint operator associated with the hessian form $H$. Now $C^+_p$ cannot be completely contained in the tangent space $D_p$ of $D$ at $p$, since otherwise the normal vector $u$ of $D$ would belong to $C^+_p$, which is impossible by construction. Hence, no neighborhood of $p$ in $W^s$ is completely contained in $D$. Thus there exists an integral curve $\gamma: [0, \infty) \to V - D$ of $\nabla f$ with $\lim_{t \to \infty} \gamma(t) = p$.

We find $0 < a < a_0$ such that $\gamma(0) \in \partial C^a$, where $C^a$ is one of the compact totally convex sets arising from the contraction of $C$ as described in Section 1. Let $t^* = \sup \{t \mid \gamma(t) \in \partial C^a\}$, $q = \gamma(t^*)$, so $\gamma(t) \in \text{int} C^a$ for all $t > t^*$, since $p \in \text{int} C^a$. Choose a supporting half-space $H \subset C_q$ for $C^a$ at $q$. Then $\exp_p(H) \supset C^a$. It follows that $\nabla f_q = \gamma(t^*) \in \bar{H}$. On the other hand, the explicit formula for the gradient and the convexity of $C^a$ obviously imply $\nabla f_q \in \bar{H}$. Thus the lemma is proved.

It is worth mentioning that the existence of the above integral curve $\gamma$ is crucial in order to bring in the convexity of the engulfing set $C^a$. Instead of using the stable manifold theorem, one can also obtain $\gamma$ by Morse Theory, but not quite so easily.

Let $C$ be compact totally convex and $S \subset C$ the soul of $C$. By induction, it follows from Lemma 3.2 that for any geodesic $c: [0,1] \to S$, the index of $c$ in $S$ is equal to the index of $c$ in $C$. This result is the essential part of the proof of Theorem 3.1 which is now a consequence of the next lemma.

**Lemma 3.3.** Let $C$ be compact totally convex and $S$ the soul of $C$. Let $p \in S$ and $u, v \in M_p$ orthonormal tangent vectors of $C$ such that $u$ is tangent to $S$ and $v$ is perpendicular to $S$ in $C$. Since $\partial S = \varnothing$, one has the geodesic $c: (-\infty, \infty) \to S$ with $c(t) = \exp_p(tu)$. If the sectional curvature $K_\sigma$ of $M$ at $p$ with respect to the plane $\sigma$ spanned by $u$ and $v$ is positive, then, for sufficiently large $t > 0$, the index of the geodesic $c|[-t, t]$ in $C$ is greater than its index in $S$.

**Proof.** The argument is fairly straightforward; compare Lemma 1 of [8]. We consider the parallel field $X$ along $c$ with $X(0) = v$ and the nonnegative function $k: R \to R$, where $k(t)$ is the curvature of $M$ with respect to the plane spanned by $X(t)$ and $\dot{c}(t)$. Set $a = k/2$. The solution of the ordinary
differential equation $\varphi'' + a\varphi = 0$ on the real line with $\varphi(0) = 1$, $\varphi'(0) = 0$, has zeros $-t_i < 0 < t_2$, $\varphi(-t_i) = \varphi(t_2) = 0$. Let $\hat{t} > \max(t_i, t_2)$ and $\psi: [-\hat{t}, \hat{t}] \to \mathbb{R}$ the function with $\psi|[-t_i, t_2] = \varphi|[-t_i, t_2]$, $\psi(t) = 0$ otherwise. Now $\psi X$ is a sectionally smooth vector field along $c|[-\hat{t}, \hat{t}]$, and we obtain for the index form in $C$,

$$I(\psi X, \psi X) = \int_{-t_1}^{t_2} (\varphi'^2 - 2\beta \varphi^3) dt < \int_{-t_2}^{t_2} (\varphi'^2 - a\varphi^3) dt = 0.$$

In order to complete the proof just observe that for any two vector fields $Y_\uparrow$, $Y_\perp$ along $c|[-\hat{t}, \hat{t}]$ in $C$, one has $I(Y_\uparrow, Y_\perp) = 0$ if $Y_\uparrow$ is tangent, and $Y_\perp$ is perpendicular to $S$ in $C$. This follows, since $S$ is totally geodesic.

We mention an application of Theorem 3.1 which was originally proved in [8].

**Corollary 3.4.** If $M$ is a complete open manifold of positive curvature, then any soul $S$ of $M$ is a point. Hence, $M$ is diffeomorphic to euclidean space $\mathbb{R}^n$.

In fact, this result can also be obtained from Theorem 1.10, even under the weaker assumption that $M$ has nonnegative curvature, but positive curvature everywhere outside some compact set.

4. Case of codim 1 and Toponogov’s Splitting Theorem

In this section we discuss the case where $\dim S = \dim M - 1$. The result we obtain is closely related to the splitting theorem of Toponogov [12] which we reprove in our context.* The splitting theorem will be applied in later sections.

**Lemma 4.1.** Let $D$ be a compact subset of a closed t.c.c.s. $C$ in an arbitrary complete riemannian manifold $M$. If the function $q \mapsto \rho(q, D)$ assumes a (relative) maximum at $p \in \text{int } C$, then for any $v \in C_p$, there is a minimal geodesic $\sigma: [0, d] \to C$ from $p$ to $D$ such that $\langle v, \dot{\sigma}(0) \rangle \geq 0$.

This is a slight generalization of a well known lemma of Berger and may be proved by precisely the same argument, see [13].

**Theorem 4.2.** Let $S \subset M$ be a soul and suppose $\dim S = \dim M - 1$. Then $\exp_S|_S(S)$ is an isometry between $S$ with its standard (flat) bundle metric and $M$.

**Proof.** By Proposition 1.3, $M = \bigcup C_\gamma$ where $C_\gamma$ is totally convex and $S = C_\gamma^{\max}$. Now for any $p \in S$ and $t > 0$ there is a minimal connection $\gamma$ to

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* In [5] we give a more general argument which shows that the splitting theorem remains true if the Ricci curvature is nonnegative.
\(\partial C\). However, since all points of \(S\) are equidistant from \(\partial C\), we have \(\gamma(0) \in \nu(S)\). But by Lemma 4.1, it then follows that \(-\gamma\) is also a minimal connection to \(\partial C\) for any \(t\). By the argument of Theorem 1.10 it follows that \(\exp_p|\nu(S)\) is a local isometry (here \(\nu(S)\) has its induced metric) and hence a covering. Since \(S\) is totally convex it follows that each \(p \in S\) has just one inverse image and hence that \(\exp_p|\nu(S)\) is a global isometry.

Recall that a line is a normal geodesic \(\sigma: (-\infty, \infty) \to M\), any segment of which is minimal.

**Theorem 4.3 (Toponogov).** \(M\) may be written uniquely as the isometric product \(\bar{M} \times R^k\), where \(\bar{M}\) contains no lines and \(R^k\) has its standard flat metric.

**Proof.** By induction, it suffices to show that if \(M\) contains a line \(\sigma\), then \(\bar{M}\) is isometric to \(\bar{M} \times R\). Now it follows from the fact that \(\sigma\) is a line that \(B_\sigma\) and \(B_{-\sigma}\) are at distance precisely 2a. By Proposition 1.3 and Lemma 1.7, any geodesic \(\tau: (-\infty, \infty) \to M\) such that \(\tau(0) = \sigma(0)\) and \(\dot{\tau}(0) \perp \dot{\sigma}(0)\), does not meet \(\text{int}((B_\sigma)')\). Hence, by Theorem 1.10, \(\rho(\tau(t), \partial B_{-\sigma}) = a\) for all \(t\) and similarly \(\rho(\tau(t), \partial B_{\sigma}) = a\). Suppose \(\rho(q, \partial((B_{-\sigma})' \cap (B_{\sigma})')) = a\) for some \(q \in (B_{-\sigma})' \cap (B_{\sigma})'\) and let \(\gamma\) be a geodesic from \(\sigma(0)\) to \(q\). Then by Proposition 1.3, \(\gamma \subset (B_{-\sigma})' \cap (B_{\sigma})'\) which implies by Lemma 1.7 that \(\dot{\gamma}(0) \perp \dot{\sigma}(0)\). Then by the above, \(\rho(q, \partial((B_{-\sigma})' \cap (B_{\sigma})')) = a\) and therefore

\[\rho(q, \partial((B_{-\sigma})' \cap (B_{\sigma})')) = a = \exp_{\partial M}(\nu(\sigma))\]

Furthermore, \(\exp_{\partial M}(\nu(\sigma))\) has no boundary. If \(q\) were a boundary point and \(q = \exp_{\partial M}(t\nu)\) then by Lemma 1.4, we would have for \(t > \tilde{t}\), that \(\exp_{\partial M}(t\nu) \notin \exp_{\partial M}(\nu(\sigma))\) which is ridiculous. The proof may now be completed as in Theorem 4.2.

The above argument may be used to show that if \(C\) is any compact t.c.s. then \(\partial C\) has at most two components and that if there are two components then \(\partial C \times [0, d]\). This implies \(M\) has either one or two ends and that if there are two ends, then \(M\) is the isometric product of \(\bar{M}\) and \(R\) with \(\bar{M}\) compact.

5. **Global behavior of geodesics**

In this section we give some results describing the global behavior of geodesics. The results provide at least a partial answer to a generalization of a problem of Chern [6]. The original problem was solved in [8].

**Theorem 5.1.** (1) If \(\gamma: (-\infty, \infty) \to M\) is a geodesic and \(\gamma \subset C\) for some

* The prime denotes set theoretic complement.
compact t.c.s. $C$, then $\gamma \subset \partial C^t$ for some $a$. Thus at least one end of $\gamma$ goes to infinity or $\gamma \subset \partial C_t$ for some $t$.

(2) Any geodesic intersecting a soul $S$ transversally, goes to infinity in both directions.

(3) If $\sigma, \tau: [0, \infty) \to M$ are geodesics, $\sigma(0) = \tau(0)$, $\langle \dot{\sigma}(0), \dot{\tau}(0) \rangle > 0$, and $\sigma$ is a ray, then $\tau$ goes to infinity.

(4) Suppose $M$ is not diffeomorphic to $\mathbb{R}^n$ and $C$ is a compact t.c.s. Then through each point $p \in C$ there is at least one geodesic $\gamma: [0, \infty) \to M$ which stays in $C$.

Theorem 5.1 is basically a consequence of Theorem 1.10.

Proof. (1) If $\gamma \subset C$ then by Theorem 1.10, $\rho(\gamma(s), \partial C)$ is a convex function defined on all of $R$ which is bounded from below. Such a function is necessarily constant. Therefore, if neither end of $\gamma$ goes to infinity then $\gamma \subset C_t$ for some $t$ and hence $\gamma \subset \partial C_t$ for some $t$.

(2) As in Section 1, we have $C_t \supset C_0 = C(1) \supset \cdots \supset C(k) = S$, with $C(i) = C(i - 1)^{\max}$. Suppose $\gamma: (-\infty, \infty) \to M$ meets $S$ transversally at $\gamma(0)$ and that say $\gamma[0, \infty) \subset C(k - 1)$. Then for $s \geq 0$ the function $\rho(\gamma(s), \partial C(k - 1))$ is convex and bounded below. Such a function must be constant. However, since $\gamma$ meets $S$ transversally, clearly this is not the case. Therefore, $\gamma[0, \infty) \nsubseteq C(k - 1)$ and the claim follows by induction.

(3) Do the basic construction at $\sigma(0) \in C_0$, (old notation). Then $\tau(0) \in C_0$, but by Lemma 1.7, $\tau \not\subset C_0$. Thus, for all $t$, $\rho(\tau(s), \partial C_t)$ is not constant and hence $\tau \not\subset C_t$ for all $t$.

(4) If $M$ is not contractible then by [10] for any $q \in C$, $\Omega_{p,q}(M)$ has non-vanishing homology groups of arbitrarily high dimension. It follows by Morse Theory that $\Omega_{p,q}$ contains a sequence of geodesics $\gamma_i \subset C$ whose lengths go to infinity. Letting $v \in M_q$ be an accumulation point of $\gamma_i/||\gamma_i||(0)$, it follows that $\gamma(s) = \exp_p sv$ has the required property.

6. Groups of isometries

We now study the isometry group $I$ of a complete manifold $M$ of non-negative curvature. Our main result is that if $I$ is not compact then $M$ contains a line. In the next section we apply this result to the study of locally homogeneous spaces.

THEOREM 6.1. Let $C$ be any compact set and let $C_t$ be the intersection of all closed totally convex sets containing $I \cdot C$. Then $C_t$ is the minimal $I$-invariant t.c.s. containing $C$ and if $C_t$ is not compact it contains a line.

Proof. The family $K$ of t.c.s. containing $I \cdot C$ is clearly invariant under $I$. Hence
\[ I(C_I) = I(\bigcap_{K \in \mathcal{K}} K) = \bigcap_{K \in \mathcal{K}} K = \bigcap_{K \in \mathcal{K}} K = C_I. \]

\( C_I \) is now clearly the minimal \( I \)-invariant t.c.s. in \( \mathcal{K} \). We claim there exists a constant \( d \) such that every point of \( C_I \) is at distance \( \leq d \) from \( I \cdot C \). Otherwise there exists a sequence \( q_i \in C_I \) and closest points \( I_i(p_i) \) with \( p_i \in C \) such that \( \rho(q_i, I_i(p_i)) \to \infty \). Then \( \rho(I_i^{-1}(q_i), p_i) = \rho(I_i^{-1}(q_i), C_i) \to \infty \). Let \( \gamma_i \) be a minimal connection from \( p_i \) to \( I_i^{-1}(q_i) \). Then by compactness of \( C \), we may assume \( p_i \to p \in C \) and \( \gamma_i \to \gamma \) where \( \gamma(0) = p \) and \( \gamma \subset C_I \) is a ray. Then \( C_I \cap (B_r)^o \) would be a t.c.s. strictly smaller than \( C_I \) which clearly contains \( I \cdot C \). Suppose \( C_I \) is not compact. Since it is convex it will contain a ray \( \sigma \) from any \( p \in C \). By what we have shown above, there exists a sequence \( I_i \in I \) such that \( \rho(I_i(\sigma(n)), C) \leq d \). Choosing a subsequence we may assume that \( I_i(\sigma(n)) \to q \). Let \( T_n \) be translation by \( n \) on \( R \) and set \( \tau_i = I_i(\sigma(T_n)) \) where \( \tau_i: (-n, n) \to C_I \). Then choosing a limit geodesic \( \tau \) such that \( \tau(0) \) is an accumulation direction of \( \tau_i(0) \), it follows that \( \tau \subset C_I \) is a line.

**Corollary 6.2.** (1) \( M \) can be written uniquely as \( \bar{M} \times R^k \) where the isometry group of \( \bar{M} \) is compact.

(2) \( I(M) = I = I(\bar{M}) \times I(R^k) \).

**Proof.** (1) This is immediate from Theorem 6.1 and Theorem 4.3, since if \( I \) is noncompact then for any \( p \), the orbit \( I \cdot p \) is noncompact.

(2) Fix \( (p, q) \in \bar{M} \times R^k \). Since any isometry \( g \) maps lines to lines and since the dimension of the space of lines through each point is precisely \( k \), any \( g \in I \) maps \( p \times R^k \) to \( g(p) \times R^k \). Then since \( \bar{M} \times q \) is perpendicular to \( p \times R^k \), \( g \) maps \( \bar{M} \times q \) to \( \bar{M} \times g(q) \). Define projections

\[ \hat{\pi}_t: I \to I(\bar{M}) \text{, } \hat{\pi}_s: I \to I(R^k) \]  

by \( \hat{\pi}_t(g)(t) = \pi_t(g(t, q)) \), \( \hat{\pi}_s(g)(s) = \pi_s(g(p, s)) \).

Then \( \hat{\pi}_t \times \hat{\pi}_s: I \to I(\bar{M}) \times I(R^k) \) is an isomorphism.

**Corollary 6.3.** In the above decomposition there is a soul of \( \bar{M} \) which is invariant under \( I(\bar{M}) \). In particular if the curvature of \( \bar{M} \) is strictly positive then there is a point \( p(=S_{\bar{M}}) \) which is invariant under \( I(\bar{M}) \).

**Proof.** Since \( I(\bar{M}) \) is compact, so is \( q_I \) for any \( q \in \bar{M} \). Then \( \partial q_I \) are also invariant. Using the procedure of Theorem 1.11 we construct a soul \( S_{\bar{M}} \) which by induction is seen to be invariant. By Corollary 3.4, if the curvature of \( \bar{M} \) is strictly positive, \( S_{\bar{M}} \) is a point.

7. Locally homogeneous spaces

In this section we specialize to the case where \( M \) is locally homogeneous. Our main result is the classification up to isometry of noncompact complete locally homogeneous manifolds of nonnegative curvature modulo the classifi-
cation up to isometry of compact locally homogeneous spaces and the action of the isometry group on the fundamental group in the compact case.* All the material of this section may be specialized to the case of flat manifolds. For some previously known results see [2], [15], [17].

**Theorem 7.1.**  (1) If $M$ is a homogeneous space of nonnegative curvature then $M$ may be written uniquely as $\tilde{M} \times \mathbb{R}^{k}$ where $\tilde{M}$ is a compact homogeneous space of nonnegative curvature.

(2) More generally, if $M$ is a complete locally homogeneous space of nonnegative curvature then there exists a compact t.c.s. $S'$, $\partial S' = \emptyset$ such that $M$ is isometric to $\nu(S')$ with its induced metric, with respect to which it is a flat vector bundle.

Alternatively, it is not hard to show that any soul $S$ of $M$ has the same property as $S'$. It is thus a consequence of Lemma 7.2 and Theorem 7.3 that $S'$ itself is a soul of $M$.

**Proof.** (1) By Theorem 4.3, $M = \tilde{M} \times \mathbb{R}^{k}$ where $\tilde{M}$ contains no lines. Hence by Theorem 6.1, $I_{\tilde{M}}$ is compact. Since $\tilde{M}$ is necessarily homogeneous, it must also be compact.

(2) If $M$ is locally homogeneous then its universal covering space $\tilde{M}$ is homogeneous and hence may be written as $\tilde{M} \times \mathbb{R}^{k}$, with $\tilde{M}$ compact. Let $\pi$ denote the group of covering transformations and let $\rho(\pi)$ denote the projection of $\pi$ on $\mathbb{R}^{k}$. By the proof of Theorem 6.1, there is a $\rho(\pi)$-invariant closed t.c.s. $C$ on which $\rho(\pi)$ acts uniformly. By induction, we may assume that $\partial C = \emptyset$ which implies that $C$ is a subspace. For if $\partial C \neq \emptyset$ then $C^{\max}$ is also $\rho(\pi)$-invariant. By choosing the origin in $\mathbb{R}^{k}$ to lie on $C$, we have $\tilde{M} = \tilde{M} \times \mathbb{R}^{l} \times \mathbb{R}^{k-l}$ with $C = \mathbb{R}^{l} \times 0$. For any $k \in \pi$ we have $k: (x, y, z) \rightarrow (\tilde{f}(x), g(y), h(z))$ with $h(0) = 0$. It follows that $M$ isometric to the flat normal bundle of the compact locally homogeneous space $S' = \tilde{M} \times \mathbb{R}^{l} \times 0/\pi$ and it is easy to see that $S'$ is totally convex.

The problem of classification up to isometry is thus reduced to classification up to isometry of flat bundles over compact spaces. Moreover, it is clearly enough to consider bundles with no parallel section, because the total space of such a bundle splits off a line isometrically. We remark that neither the proof of part (2) of Theorem 7.1 nor the classification given below makes essential use of nonnegative curvature. Thus what we are really doing is classifying spaces whose universal covering splits as $\tilde{M} \times \mathbb{R}^{k}$ with $\tilde{M}$ compact, modulo compact spaces of this type.

* In [5] we show that these results remain valid for locally homogeneous spaces of nonnegative Ricci curvature (see the remark after Theorem 7.1).
Lemma 7.2. Let \( M \) be locally homogeneous and write \( M = \nu(S') \) as in Theorem 7.1.

(1) If \( p \in M \), then all rays from \( p \) are contained in the fibre through \( p \).

(2) Any geodesic \( \perp S' \) is a ray.

Proof. (1) Let \( \gamma: [0, \infty) \to M \) be any normal geodesic, \( \gamma(0) = p \), and assume \( \gamma \) is not contained in the fibre through \( p \). Then using the local product structure it follows that \( \gamma \) makes constant angle \( \alpha \) with all fibres and hence its projection on \( S' \), \( \hat{\gamma} \), is an infinite geodesic in \( S' \). Let \( \iota(S') \) denote the injectivity radius of the exponential map on \( S' \). It follows by compactness of \( S' \) that given \( 0 < \varepsilon < (1 - \sin \alpha) \cdot \iota(S') \cdot \sec \alpha \) there exist parameter values \( t_i < t'_i \) such that \( \rho(\hat{\gamma}(t_i), \hat{\gamma}(t'_i)) < \varepsilon \). However, \( t'_i - t_i > \iota(S') \cdot \sec \alpha \).

Using the local product structure, we see that there is a broken geodesic between \( \gamma(t_i) \) and \( \gamma(t'_i) \) of length \( (t'_i - t_i) \cdot \sin \alpha + \varepsilon < t'_i - t_i \). But since \( t'_i - t_i \) is just the arc length of \( \gamma \), we conclude that \( \gamma \) is not a ray.

(2) Let \( \sigma \) be any geodesic \( \perp S' \). Given \( t \) let \( \tau \) be a minimal connection from \( \sigma(t) \) to \( S' \). Then \( \tau \perp S' \). But since, as we have seen, \( \exp \mid \nu(S') \mid = 1 - 1 \) we have necessarily that \( \tau = \sigma \). Since any segment of \( \sigma \) is a minimal connection to \( S' \) it is certainly minimal between its end points.

Theorem 7.3. Let \( M \) be locally homogeneous. Then the soul of \( M \) is unique if and only if \( M \) does not contain a line.

Proof. Let \( M = \nu(S') \) and let \( \hat{S} \) be any other t.c.s. with \( \partial \hat{S} = \emptyset \) such that \( M = \nu(\hat{S}) \). At each \( p \in \hat{S} \) all geodesics \( \perp \hat{S} \) are rays and hence are contained in the fibre of \( \nu(S') \) through \( p \). By comparing dimensions we see that the tangent space to \( \hat{S} \) is precisely the orthogonal complement of the fibre of \( \nu(S') \). Using the local isometric splitting of \( \nu(S') \) it follows that if \( \tau: \nu(S') \to S' \) is the projection, then \( \tau \mid \hat{S} \) is a riemannian covering. Moreover, locally \( \hat{S} \) defines a parallel section of \( \nu(S') \). Finally, since \( \hat{S} \) is totally convex, it can not intersect a given fibre in two distinct points \( p \) and \( q \). Otherwise it would contain the segment in the fibre connecting them, but this contradicts the description of the tangent space of \( \nu(S') \) given above. We conclude that \( \hat{S} \) defines a global parallel section of \( \nu(S') \) and hence that \( M \) contains a line.

In order to state the classification theorem we need some preliminary definitions. Given any riemannian manifold \( S \) and \( p \in S \), there is a natural representation \( \Gamma \) of the isometry group \( I \) on the group \( \text{Out}_S \) of outer automorphisms of the fundamental group \( \pi \) at \( p \). This is defined as follows: For each \( g \in I \) choose a path \( \varphi \) from \( p \) to \( g(p) \). For each \( \{h\} \in \pi \) choose a representative \( h \). Then define \( \Gamma(g) \) to be the residue class of the automorphism \( \{h\} \to \{-\varphi \cdot g(h) \cdot \varphi \} \). It is trivial to check that this is independent of \( \varphi \). Let \( PC_\pi \)
denote the group of all permutations of the set $C_x$ of conjugacy classes, of orthogonal representations not containing a trivial summand, of $\pi$ on some fixed euclidean vector space $V$. Then there is a natural representation $\Delta$ of $\Out_x$ on $PC_x$ defined as follows: Let $r$ be a representative for $[r] \in C_x$ and $s$ a representative of $[s] \in \Out_x$. Then define $\Delta([s])([r])$ to be the conjugacy class of $r \circ s$.

**Theorem 7.4 (Classification).** Let $S$ be a compact locally homogeneous space of nonnegative curvature with isometry group $I$ and fundamental group $\pi$ at $p$. Let $V$ be a fixed euclidean vector space. Then the isometry classes of irreducible noncompact complete locally homogeneous spaces of nonnegative curvature with soul isometric to $S$ and fibre of dimension equal to $\dim V$ are in natural bijective correspondence with the elements of $\Sym_x / \Delta \circ \Gamma(I)$.

**Proof.** The correspondence is defined as follows: Given $M$ choose some isometry $J$ of $S_M$ with $S$ and some isometry $A$ of the fibre of $\nu(S_M)$ at $J^{-1}(p)$ with $V$. Then if $P$ denotes parallel translation in $\nu(S_M)$ along curves in $S_M$, define a representation $r_M$ of $\pi$ on $V$ by $r((h)) = A \circ P_{J^{-1}(h)} \circ A^{-1}$. We send $M$ to the equivalence class $\{[r_M]\}$ under the action $\Delta \circ \Gamma(I)$, of the conjugacy class of this representation.

1. The correspondence $M \rightarrow \{[r_M]\}$ is well defined: Let $J_i, A_i$ be different isometries. Then

$$r_i((h)) = A_i \circ P_{J_i^{-1}(h)} \circ A_i^{-1}.$$ 

Choose $\varphi$ from $p$ to $JJ_i^{-1}(p)$. Then

$$r_i((h)) = (A_i \circ P_{J_i^{-1}(p)} \circ A_i^{-1}) \circ A \circ P_{J_i^{-1}(p)} \circ P_{J_i^{-1}(J_i \circ J_i^{-1}(h))} \circ P_{J_i^{-1}(p)} \circ A^{-1} \circ (A \circ J_i^{-1} \circ A_i^{-1}).$$

Thus $r_i$ is conjugate under $(A_i \circ P_{J_i^{-1}(p)} \circ A_i^{-1})$ to $\Delta \circ \Gamma(J \circ J_i^{-1})r$, which shows that the correspondence is well defined.

2. Given any representation $r$ of $\pi$ on $V$ one may construct by a well known method, a flat vector bundle over $S$ whose fibre at $p$ is naturally identified with $V$ and whose holonomy representation is $r$. Thus the above correspondence is clearly onto.

3. Finally let $M_i$ be another manifold which is isometric to $M$, $K: M_i \rightarrow M$. By Theorem 7.3 necessarily $K(S_{M_i}) = S_M$. Then since $K$ is an isometry, it must map fibres of $\nu(S_{M_i})$ to fibres of $\nu(S_M)$ and in particular it induces an isometry of fibres at $J^{-1}(p)$ and $K^{-1} \circ J^{-1}(p)$ which commutes with the holonomy representations on these fibres. Then $J$ and $J \circ K$ will induce exactly the same representations of $\pi$ and hence it follows that $M_i$ isometric to $M$ implies that $\{[r_{M_i}]\} = \{[r_M]\}$. Conversely if $\{[r_{M_i}]\} = \{[r_M]\}$ then by definition
there exist isometries $J_{2}$, $A_{2}$ and $B$: $V \to V$, $I$: $S \to S$ and a curve $\varphi$ from $p$ to $I(p)$ such that for any $h$,
\[
B \circ A_{2} \circ P_{J_{2}^{-1}(\varphi \cdot I_{I(p)\varphi})} \circ A_{2}^{-1} \circ B^{-1} = A \circ P_{J_{2}^{-1}(h)} \circ A^{-1}.
\]
Then
\[
A^{-1} \circ B \circ A_{2} \circ P_{J_{2}^{-1}(\varphi)} \circ P_{J_{2}^{-1}(I \cdot J_{2}^{-1}(h))} \circ P_{J_{2}^{-1}(\varphi)} \circ A_{2}^{-1} \circ B^{-1} \circ A = P_{J_{2}^{-1}(h)}.
\]
Set $J_{2}^{-1}(p) = q$, set $P_{J_{2}^{-1}(\varphi)} \circ A_{2}^{-1} \circ B^{-1} \circ A = C$, $C$: $\text{Fib}_{q} \to \text{Fib}_{J_{2}^{-1}(I \cdot J_{2}^{-1}(q))}$, and $J_{2}^{-1}(h) = k$. Then we have isometries $J_{2}^{-1} \circ I \circ J$: $S_{M} \to S_{M_{2}}$ and $C$ such that:
\[
C^{-1} \circ P_{J_{2}^{-1}(I \cdot J_{2}^{-1}(h))} \circ C = P_{k}
\]
for any $k$. It follows easily that $J_{2}^{-1} \circ I \circ J$ extends to an isometry $M \to M_{2}$.

8. Classification in dim $\leq 3$

The results of the previous sections allow us to classify up to isometry, the complete noncompact manifolds of nonnegative curvature in dim $\leq 3$. Let $P_{2}$ (resp. $P_{3}$) denote $\mathbb{R}^{2}$ (resp. $\mathbb{R}^{3}$) together with a complete metric of nonnegative curvature for which $P_{2}$ (resp. $P_{3}$) contains no lines. Then a classical theorem of Cohn-Vossen says: $P_{2}$, the flat cylinder, and the nontrivial line bundle over $S^{1}$ (Moebius band) are the only complete noncompact two-dimensional manifolds of nonnegative curvature.

**Theorem 8.1.** The isometry classes of noncompact complete 3-dimensional manifolds of nonnegative curvature are as follows:

<table>
<thead>
<tr>
<th>dim $S$</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$#$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>(a) $P_{3}$,</td>
</tr>
<tr>
<td>$\infty \times 1$</td>
<td>(b) $P_{2} \times \mathbb{R}$,</td>
</tr>
<tr>
<td>$1$</td>
<td>(c) $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dim $S$</th>
<th>Type</th>
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</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$#$</td>
</tr>
<tr>
<td>$\infty \times \infty$</td>
<td>(a) $P_{2}$-bundles over $S^{1}(r)$ obtained as follows: Take some $P_{2}$ and a representation $\varphi$: $Z \to I_{p} \subset O(2)$ where $I_{p}$ is the isotropy group at some point $p \in P_{2}$ which is fixed by $I(P_{2})$. Let $T_{r}$ be the representation of $Z$ which takes 1 into translation by $r$ on $\mathbb{R}$. Then $M = \mathbb{R} \times P_{2}(a \to T_{r}(a) \times \varphi(a))$. Distinct representations give distinct bundles;</td>
</tr>
<tr>
<td>$\infty \times \infty$</td>
<td>(b) $P_{3}$-bundles over $S^{1}(r)$ formed as above. (See Section 7.)</td>
</tr>
<tr>
<td>$2$</td>
<td>$#$</td>
</tr>
</tbody>
</table>

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$\infty \times 1$  (a) $S^1 \times \mathbb{R}$,
$\infty \times 2$  (b) trivial or nontrivial line bundle over some $RP(2)$ (real projective space),
$\infty \times \infty \times 3$  (c) flat line bundles over some flat torus,
$\infty \times \infty \times 4$  (d) flat line bundles over some flat klein bottle.

**Proof.** These exhaust the possible dimensions for $S$. The case $\dim S = 0$ is trivial. The case $\dim S = 2$ follows from Theorem 4.2. In case $\dim S = 1$, then by Theorem 4.3, $M = M \times \mathbb{R}$, $M = \mathbb{R} \times \mathbb{R}$ or $P_2$. The stated classification follows by an argument similar to Theorem 7.1.

9. Coverings and the fundamental group

Now let $M$ be compact, $\pi = \pi_\circ(M, p)$ the fundamental group of $M$ with respect to some fixed point $p \in M$, and $\tilde{M} \to M$ the universal riemannian covering. If $I$ denotes the isometry group of $\tilde{M}$, there is a canonical representation $\pi \subset I$ as discrete subgroup, and $\pi$ acts on $\tilde{M}$ as the group of covering transformations, $\tilde{M}/\pi = M$.

**Theorem 9.1.** There exists a compact simply connected $(n-k)$-dimensional manifold $M_0$ of nonnegative curvature, $0 \leq k \leq n = \dim M$, such that $\tilde{M}$ is isometrically diffeomorphic with the riemannian product $M_0 \times \mathbb{R}^k$ of $M_0$ and flat euclidean space $\mathbb{R}^k$.

**Proof.** By Corollary 7.2 we have an isometric splitting $\tilde{M} \cong M_0 \times \mathbb{R}^k$ and a decomposition $I \cong I_0 \times E$ as direct product, where $I_0$ is the compact isometry group of $M_0$ and $E$ the group of euclidean motions in $\mathbb{R}^k$. Since $M$ is compact, we find compact metric balls $B \subset M_0$, $B' \subset \mathbb{R}^k$ such that $\pi(B \times B') = \tilde{M}$. Now $\pi(B \times B') \subset I(B \times B') = I_0(B) \times E(B') = I_0(B) \times \mathbb{R}^k = \tilde{M}$, hence $I_0(B) = M_0$ is compact.

**Theorem 9.2.** There is a diagram of covering maps where the vertical maps are isometric coverings, the horizontal maps locally isometrically trivial fibrations, and the diagonal map is a diffeomorphism. $T^k$ is a flat torus.

\[
\begin{array}{cccccc}
M_0 & \to & \tilde{M} \cong M_0 \times \mathbb{R}^k & \to & \mathbb{R}^k \\
\downarrow & & \downarrow & & \downarrow \\
M_1 & \to & \tilde{M} & \to & T^k \\
\downarrow & & \downarrow & & \downarrow \\
M & & M_1 \times T^k & &
\end{array}
\]

**Proof.** The kernel of the natural projection $\varphi: \pi \subset I_0 \times E \to E$ is finite.
since $I_0$ compact. Consider the riemannian covering $\tilde{M} \cong M_0 \times R^k \to M^* = \tilde{M}/\ker \varphi \cong M_1 \times R^k$ and the corresponding isometry group $I^* \cong I_1 \times E$ of $M_1 \times R^k$, $I$, compact. The projection $\varphi^* : I_1 \times E \to E$ maps $\pi^* = \pi/\ker \varphi \subset I^*$ isomorphically onto a discrete uniform subgroup of $E$, notice that $M_1 \times R^k \to R^k$ induces a continuous map $M = M^*/\pi^* \to R^k/\varphi^*\pi^*$, hence $R^k/\varphi^*\pi^*$ compact. Therefore, by the Bieberbach Theorem, $\pi^*$ contains a normal free abelian subgroup $\Gamma^*$ of finite index, $rk\Gamma^* = k$, where $\varphi^*\Gamma^*$ is a lattice in the subgroup $L \cong R^k$ of translations in $E$. We have the compact abelian subgroup $H = \tilde{\psi}\Gamma^*$ of $I_0$, $\tilde{\psi} : I_1 \times E \to I_1$ projection, and the toral connected component $H_1$ of the identity in $H$. Now $\Gamma = \Gamma^* \cap \psi^{-1}H_1$ is invariant and of finite index in $\pi^*$.

We claim $\tilde{M} = M_1 \times R^k/\Gamma$ is diffeomorphic to $M_1 \times T^k$. Extend $\psi|\Gamma$ to a Lie group homomorphism $\tilde{\psi} : L \to H_1$, $\tilde{\psi}\varphi^*|\Gamma = \psi|\Gamma$. We define a diffeomorphism $f : M_1 \times L \to M_1 \times R^k \cong M^*$, $f(p, a) = (\tilde{\psi}(a) \cdot p, a)$. Then $f$ is $\Gamma$-equivariant, where $\Gamma$ acts canonically on $M_1 \times L$, $(p, a) \to (p, \varphi^*(g) \cdot a)$, so $f$ induces a diffeomorphism $\tilde{f} : M_1 \times L/\Gamma = M_1 \times T^k \to M^*/\Gamma$, as claimed.

Since the action of $\Gamma$ respects the decomposition $M_1 \times R^k$ and $\varphi^*(\Gamma)$ acts freely on $R^k$, the natural map $M_1 \times R^k/\Gamma \to R^k/\varphi^*\Gamma = T^k$ is easily seen to be a locally isometrically trivial fibration.

The diffeomorphism in the last theorem can be chosen to be isometric (product metric on $M_1 \times T^k$ for the same flat torus $T^k$) if for example $I_0$ is discrete. However, this is not possible in general. When $\dim I_0 > 0$, the action of $\pi$ on $\tilde{M} = M_0 \times R^k$ may be twisted. A simple example is furnished by $\tilde{M} = S^1 \times R$ (natural product metric), where the integers $\pi = Z$ act on $\tilde{M}$ by glide rotations $(p, a) \to (r^mp, a + m)$, where $m \in Z$ and $r$ is a rotation of $S^1$ through an angle $\alpha$ about some axis with $\alpha/2\pi$ irrational. Then $\tilde{M}/Z$ is diffeomorphic with $S^1 \times S^1$, but no covering of $\tilde{M}/Z$ is isometric with some riemannian product $S^1 \times S^1$.

We see from the proof of Theorem 9.2 that the quotient $\pi^*$ of the fundamental group $\pi$ of $M$ by a finite normal subgroup is isomorphic to a crystallographic group (discrete uniform subgroup of a euclidean group). Such a group is always an extension of a finite group by a lattice, [17]. This generalizes in our situation the classical theorem of Myers for compact manifolds of positive mean curvature, saying that $\pi$ has to be finite, as well as a result of Milnor in [9] on the growth of $\pi$. In [5], we will show that the Theorems 9.1, 9.2 and the Corollaries 9.4, 9.5 remain valid under the assumption that $M$ has nonnegative Ricci curvature. Combining the above conclusions with Theorem 2.1, we can now describe the structure of $\pi$ in the general case,
where $M$ need not be compact anymore.

**Theorem 9.3.** Let $M$ be complete and let $S$ the soul of $M$. Then $\pi = \pi_1(M) \cong \pi_1(S)$. There exists an invariant finite subgroup $\Phi \subset \pi$ such that $\pi^*_\pi = \pi/\Phi$ is isomorphic to a crystallographic group, so $\pi^*_\pi$ contains a free abelian normal subgroup $\Gamma \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ of rank $k$, $0 \leq k \leq \dim S \leq \dim M$, $\pi^*/\Gamma$ finite. In particular, $M$ is a compact flat manifold if $k = \dim M$.

**Corollary 9.4.** When $\pi$ is infinite, then the Euler characteristic $\chi(M)$ is zero. If in addition $M$ is compact, all real Pontrjagin numbers vanish.

**Proof.** By Theorem 9.2 a finite covering $\hat{S}$ of $S$ is diffeomorphic to a product of a compact manifold and a torus of positive dimension, since $\pi$ infinite. Hence $\chi(\hat{S}) = 0$ and so $\chi(M) = \chi(S) = 0$. Since $\hat{S}$ contains a torus factor which bounds, $\hat{S}$ itself bounds. Hence all Pontrjagin numbers of $\hat{S}$ vanish. By naturality, the same is true for $S = M$.

The following result gives a more general answer to a conjecture of Auslander and Wolf posed for the torus in [16].

**Corollary 9.5.** Let $M$ be a compact manifold of nonnegative curvature. If $M$ is a space of type $K(\pi, 1)$, then $M$ is a flat manifold, in particular the fundamental group $\pi$ has to be a Bieberbach group.

**Proof.** The universal riemannian covering $\check{M}$ of $M$ is contractible, so $\check{M}$ is isometric to flat euclidean space $\mathbb{R}^*$ by Theorem 9.2.

10. Some examples and problems

We conclude by describing very briefly the construction of a certain class of interesting complete manifolds of nonnegative curvature. Most of these manifolds are not locally homogeneous and have no local product structure (compare Section 3). Nonetheless, they still seem to have many plane sections with zero curvature.

Let $N$ be a complete riemannian manifold and $H$ a group of isometries acting freely on $N$ so that the orbit space $M = N/H$ is a manifold and basis of a principal fibration $H \to N \to M$ with natural projection $\pi$. Since $H$ acts by isometries, the metric of $N$ projects down to a complete metric for $M$ with respect to which $\pi$ becomes a riemannian submersion. It is known that $\pi$ is curvature nondecreasing. More precisely,

$$K_N(\pi_* X, \pi_* Y) = K_N(X, Y) + \frac{3}{4}|| [X, Y]^\ast ||^2,$$

where $X, Y$ are orthonormal vector fields on $N$ which are horizontal (orthogonal to orbits) and $Z^\ast$ denotes the vertical (fibre tangential) component of a

As a special case, consider a Lie group $G$ with biinvariant metric and a closed subgroup $H$. Any (not necessarily free) action of $H$ on a complete manifold $N$ with $K \geq 0$ induces a free diagonal action of $H$ on the riemannian product $G \times N$ (translation on the first factor). Then the above yields a complete quotient manifold $M = G \times N/H$ which is nonnegatively curved. The obvious fibration $N \to M \to G/H$ is associated to the principal bundle $H \to G \to G/H$.

For example, let $G = SO(n+1)$ and $H = SO(n)$ acting on flat euclidean space $N = \mathbb{R}^n$ by rotation. Then, $M = G \times N/H$ is the tangent bundle $TS^*$ of the sphere $S^*$. Hence, $M = TS^*$ admits complete metrics with $K \geq 0$. It turns out that the soul of $M$ is unique for $n \geq 2$, $S = \pi(G \times 0)$. There is a fibration $F \to M \to S$, the fibers $\pi(g \times N)$ for $g \in G$ being totally geodesic in $M$ and all isometric to $F = \pi(e \times N)$. The fibers are not flat.

We have touched various open questions in this paper. Let us finally single out two problems that seem to be of particular interest and significance.

(1) Suppose, $M$ is complete and noncompact with $K \geq 0$, but $K > 0$ at some point. Is then the soul of $M$ always a point, or equivalently, is $M$ diffeomorphic to euclidean space $\mathbb{R}^n$? (This is known to be true for immersed hypersurfaces in euclidean space.)

(2) Do all vector bundles over the standard euclidean $n$-sphere admit complete metrics with $K \geq 0$?

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**Bibliography**


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