and the theorem follows immediately in view of the uniform estimates in Proposition 2.3.

Suppose the initial hypersurface M_0 satisfies the equation $H_0 = \langle \mathbf{F}_0, \nu \rangle$. Then the homothetic deformation

$$\mathbf{F}(p,t) = (2(T-t))^{1/2}\mathbf{F}(p,0)$$

satisfies

$$\left(\frac{d}{dt}\mathbf{F}(p,t)\right)^{\perp} = \frac{1}{(2(T-t))^{1/2}} H_0 \cdot \nu = -H\nu.$$

So up to a tangential deformation the mean curvature flow is realized by homotheties for these initial surfaces.

Then Theorem 3.5 states that singularities of the mean curvature flow satisfying the growth rate estimate (2) are asymptotically selfsimilar.

Natural open questions are concerned with the uniqueness of the limit in Proposition 3.4 and the number of solutions to equation (6). In the last two sections we address these questions in some special cases.

4. Surfaces of positive mean curvature

If the mean curvature H is positive on the initial hypersurface M_0, it will stay positive on M_t as long as a solution of (1) exists (see [7, Corollary 3.5(i)]). Thus it is natural to try to classify solutions of the selfsimilarity condition (6) in this special case. In case $n = 1$ it was shown by Abresch and Langer in [1] that there is a 2-parameter family of closed immersed curves in $\mathbb{R}^2$ of positive geodesic curvature which are selfsimilar solutions of (1). We prove that in higher dimensions the sphere is the only compact hypersurface of positive mean curvature moving under selfsimilarities.

4.1 Theorem. *If M^n, $n \geq 2$, is compact with nonnegative mean curvature H and satisfies the equation $H = \langle \mathbf{x}, \nu \rangle$, then M^n is a sphere of radius $\sqrt{n}$.*

Proof. We differentiate the equation $H = \langle \mathbf{x}, \nu \rangle$ in an orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ on M^n and obtain

$$\nabla_i H = \langle \mathbf{x}, \mathbf{e}_l \rangle h_{li}, \tag{7}$$

$$\nabla_i \nabla_j H = h_{ij} - H h_{il} h_{lj} + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l h_{ij}. \tag{8}$$

Here we used again $H = \langle \mathbf{x}, \nu \rangle$ and the Codazzi equation. Contracting now (8) with g_{ij} and h_{ij} respectively we derive

$$\Delta H = H - H|A|^2 + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l H, \tag{9}$$

$$h_{ij} \nabla_i \nabla_j H = |A|^2 - H\,\mathrm{tr}(A^3) + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l h_{ij} \cdot h_{ij}. \tag{10}$$

Simons' identity states that (see e.g. [7, Lemma 2.1])

$$\Delta|A|^2 = 2h_{ij}\nabla_i\nabla_j H + 2|\nabla A|^2 + 2H\,\mathrm{tr}(A^3) - 2|A|^4. \tag{11}$$

So we derive from (10) that

$$\Delta|A|^2 = 2|\nabla A|^2 + 2|A|^2 - 2|A|^4 + 2\langle \mathbf{x}, \mathbf{e}_l\rangle \nabla_l h_{ij} h_{ij}.$$

Now notice that in view of (9) and the maximum principle, H satisfies the strict inequality $H > 0$. We are then ready to compute

$$\Delta\left(\frac{|A|^2}{H^2}\right) = \frac{\Delta|A|^2}{H^2} - \frac{2|A|^2}{H^3}\Delta H - \frac{4}{H^3}\nabla_i|A|^2\nabla_i H + \frac{6|A|^2}{H^4}|\nabla H|^2.$$

From (9) and (11) we obtain

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^2}\right) = \frac{2}{H^4}\Big(&H^2|\nabla A|^2 + \frac{1}{2}\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l|A|^2H^2 \\ &- H|A|^2\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l H - 2H\nabla_i|A|^2\nabla_i H + 3|A|^2|\nabla H|^2\Big).\end{aligned}$$

The right-hand side thus equals

$$\begin{aligned}\frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 + \frac{2}{H^4}\Big\{&2|A|^2|\nabla H|^2 - H\nabla_i|A|^2\nabla_i H \\ &+\frac{1}{2}H^2\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l|A|^2 - |A|^2\langle \mathbf{x}, \mathbf{e}_l\rangle H\nabla_l H\Big\}\end{aligned}$$

since

$$|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 = |A|^2|\nabla H|^2 + |\nabla A|^2 H^2 - H\nabla_l H\nabla_l|A|^2.$$

Now notice that

$$\nabla_i\left(\frac{|A|^2}{H^2}\right) = \frac{1}{H^2}\nabla_i|A|^2 - \frac{2|A|^2}{H^3}\nabla_i H,$$

such that finally

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^2}\right) &= \frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 \\ &\quad - \frac{2}{H}\nabla_i H\nabla_i\left(\frac{|A|^2}{H^2}\right) + \langle \mathbf{x}, \mathbf{e}_i\rangle\nabla_i\left(\frac{|A|^2}{H^2}\right).\end{aligned} \tag{12}$$

Since M is compact, the maximum principle then implies that $|A|^2 = \alpha H^2$ with a fixed constant α and also

$$|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 \equiv 0 \quad \text{on } M^n. \tag{13}$$

We split the tensor $h_{ij}\nabla_l H - \nabla_l h_{ij} H$ into its symmetric and antisymmetric parts and obtain from (13) and Codazzi's equation

$$|h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 \equiv 0. \tag{14}$$

At a given point of M^n we now rotate $\mathbf{e}_1, \cdots, \mathbf{e}_n$ such that $\mathbf{e}_1 = \nabla H / |\nabla H|$ points in the direction of the gradient of the mean curvature; then

$$0 = |h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 = 2|\nabla H|^2\left(|A|^2 - \sum_{i=1}^{n} h_{1i}^2\right).$$

Thus at each point of M^n we have either $|\nabla H|^2 = 0$ or $|A|^2 = \sum_{i=1}^n h_{1i}^2$. If $|\nabla H|^2 \equiv 0$ it follows immediately that M^n is a sphere and we are done.

So suppose there is a point in M where $|A|^2 = \sum_{i=1}^n h_{1i}^2$. Since

$$|A|^2 = h_{11}^2 + 2\sum_{i=1}^{n} h_{1i}^2 + \sum_{i,j\neq 1}^{n} h_{ij}^2,$$

this is only possible if $h_{ij} = 0$ unless $i = j = 1$. Then we have $|A|^2 = H^2$ at this point and therefore everywhere on M. Now we integrate (9) and obtain after integration by parts

$$\begin{aligned}\int_M H^3\,d\mu &= \int_M H\,d\mu + \int_M \langle \mathbf{x}, \mathbf{e}_l\rangle \nabla_l H\,d\mu \\ &= \int_M H\,d\mu - n\int_M H\,d\mu + \int_M \langle \mathbf{x}, \nu\rangle H^2\,d\mu.\end{aligned}$$

Since $\langle \mathbf{x}, \nu\rangle = H$, we derive $(n-1)\int_M H\,d\mu = 0$, which is a contradiction for $n \geq 2$. This completes the proof of Theorem 4.1.

Remarks. (i) The assumption $H \geq 0$ seems to be necessary: The author was told by M. Grayson that there is numerical evidence for the existence of an imbedded torus in $\mathbb{R}^3$ satisfying (6). (ii) In the noncompact case we expect for $n = 2$ cylinders to be the only imbedded surfaces satisfying (6) (see also §5).

5. The rotationally symmetric shrinking neck

In this section we consider a two-dimensional rotationally symmetric hypersurface M_0 with positive mean curvature. We prove that in this case all singularities satisfy the natural blow-up estimate (2) and behave asymptotically like cylinders. The rotationally symmetric case was first studied by R. Hamilton (oral communication), who observed Lemmas 5.1 and 5.2 of this section.

Let $y_0 \colon [a, b] \to \mathbb{R}$ be a smooth positive function on the bounded interval $[a, b]$ with $y_0'(a) = y_0'(b) = 0$, and consider the 2-dimensional hypersurface M_0 in $\mathbb{R}^3$ generated by rotating graph y around the x_1-axis. To compute