1

Definition and Short Time Existence

1.1 Notations and Preliminaries

In this section we introduce some basic notations and facts about Riemannian manifolds and their submanifolds, a good reference is [50].

In all the lectures the convention of summing over the repeated indices will be adopted.

The main objects we will consider are n–dimensional, complete hypersurfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs (M, φ) where M is an n–dimensional, smooth manifold with empty boundary and $\varphi : M \to \mathbb{R}^{n+1}$ is a smooth immersion (the rank of the differential $d\varphi$ is equal to n everywhere on M).

The manifold M gets in a natural way a metric tensor g turning it into a Riemannian manifold (M, g) by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map φ.

Taking local coordinates around $p \in M$, we have local bases of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and 1–forms $\{dx_j\}$.

We will denote the vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, the 1–forms by $\omega = \omega_j$, that is, $\omega = \omega_j dx_j$ and a general mixed tensor by $T = T^{i_1...i_k}_{j_1...j_l}$, where the indices refer to the local basis.

Sometimes we will consider tensors along M viewing it as a submanifold of $\mathbb{R}^{n+1}$ via the map φ, in such case we will use the Greek indices to denote the components of the tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^{n+1}$, for instance, given a vector field X along M, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

The metric g of M extended to tensors is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1...i_k}_{j_1...j_l} S^{s_1...s_k}_{z_1...z_l} \,,$$

where g_{ij} is the matrix of the coefficients of g in local coordinates and g^{ij} is its inverse matrix. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T, T)} \,.$$

The scalar product of $\mathbb{R}^{n+1}$ will be denoted by $\langle \cdot \,|\, \cdot \rangle$. As the metric g is obtained by pulling it back via φ, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (d\varphi^* \langle \cdot \,|\, \cdot \rangle)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial \varphi}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle \,.$$

The canonical measure induced by the metric g is given in a coordinate chart by $\mu = \sqrt{G}\, \mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivative on (M, g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k\,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji}\omega_k\,,$$

where the Christoffel symbols Γ^i_{jk} are expressed by the formula,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\right).$$

The covariant derivative ∇T of a general tensor $T = T^{i_1 \ldots i_k}_{j_1 \ldots j_l}$ will be denoted by $\nabla_s T^{i_1 \ldots i_k}_{j_1 \ldots j_l} = (\nabla T)^{i_1 \ldots i_k}_{s j_1 \ldots j_l}$ (we recall that such extension of the covariant derivative is uniquely defined on the full tensor algebra by imposing the Leibniz rule and the commutativity with any metric contraction).
$\nabla^m T$ will stand for the m–th iterated covariant derivative of T.

The gradient ∇f of a function and the divergence $\operatorname{div} X$ of a vector field at a point $p \in M$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_p M$$

and

$$\operatorname{div} X = \operatorname{tr} \nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k\,.$$

The (rough) Laplacian ΔT of a tensor T is given by

$$\Delta T = g^{ij} \nabla_i \nabla_j T\,.$$

If X is a smooth vector field with compact support on M, as $\partial M = \emptyset$ the following *divergence theorem* holds

$$\int_M \operatorname{div} X \, d\mu = 0\,,$$

which clearly implies, in particular,

$$\int_M \Delta f \, d\mu = 0$$

for every smooth function $f : M \to \mathbb{R}$ with compact support.

Since φ is locally an embedding in $\mathbb{R}^{n+1}$, at every point $p \in M$ we can define up to a sign a unit normal vector $\nu(p)$. Locally, we can always choose ν in order that it is smooth.
If the hypersurface M is compact and embedded, that is, the map φ is one–to-one, the *inside* of M is easily defined and we will consider ν to be the *inner pointing* unit normal vector at every point of M. In this case the vector field $\nu : M \to \mathbb{R}^{n+1}$ is globally smooth.

The *second fundamental form* $\mathrm{A} = h_{ij}$ of M is the symmetric 2–form defined as follows,

$$h_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle$$

and the *mean curvature* H is the trace of A, that is $\mathrm{H} = g^{ij} h_{ij}$. Despite its name, H is the *sum* of the eigenvalues of the second fundamental form, not their average mean (some few authors actually define H/n as the mean curvature).

Remark 1.1.1. Notice that since the unit normal ν is defined up to a sign, the same is true for A and H. Instead, the *vector valued second fundamental form* $h_{ij}\nu$, which is a 2–form with values in $\mathbb{R}^{n+1}$, and the *mean curvature vector* $\mathrm{H}\nu$ are uniquely defined.
With our choice of ν as the inner pointing unit normal, the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a positive definite second fundamental form and positive mean curvature and the same holds for every strictly *convex* hypersurface of $\mathbb{R}^{n+1}$.

We advise the reader that, by a little abuse of terminology, we will always say that a hypersurface is convex when its second fundamental form is nonnegative definite, strictly convex when it is positive definite. If the hypersurface is embedded, convexity in such sense is equivalent to the usual definition that the hypersurface bounds a convex subset of the Euclidean space.

The linear map $W_p : T_pM \to T_pM$ given by $W_p(v) = h^i_j(p)v^j \frac{\partial}{\partial x_i}$ is called the *Weingarten operator* and its eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ the *principal curvatures* at the point $p \in M$. It is easy to see that $\mathrm{H} = \lambda_1 + \cdots + \lambda_n$ and $|\mathrm{A}|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.

Exercise 1.1.2. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $\varphi(x) = (x, f(x))$, we have

$$g_{ij} = \delta_{ij} + f_i f_j\,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{(\sqrt{1 + |\nabla f|^2})^3} = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)$$

where $f_i = \partial_i f$ and $\mathrm{Hess} f$ is the Hessian of the function f.

Exercise 1.1.3. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the *zero set* of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, with $\nabla f \neq 0$ on such level set, we have

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right)\,.$$

The following Gauss–Weingarten relations will be fundamental,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu\,, \qquad \frac{\partial \nu}{\partial x_j} = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x_s}\,. \tag{1.1.1}$$

Actually, they express the fact that $\nabla^M = \nabla^{\mathbb{R}^{n+1}} - \mathrm{A}\nu$. We recall that considering M locally as a regular submanifold of $\mathbb{R}^{n+1}$, we have $\nabla^M_X Y = (\nabla^{\mathbb{R}^{n+1}}_X \widetilde{Y})^M$ where the sign ^{M} denotes the projection on the tangent space to M and $\widetilde{Y}$ is a local extension of the field Y in a local neighborhood $\Omega \subset \mathbb{R}^{n+1}$ of $\varphi(M)$.
Notice that, by these relations, it follows

$$\Delta\varphi = g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k}\right) = g^{ij} h_{ij}\nu = \mathrm{H}\nu\,. \tag{1.1.2}$$

By straightforward computations, we can see that the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed by means of the second fundamental form as follows,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g\Big(\nabla^2_{ji}\frac{\partial}{\partial x_k} - \nabla^2_{ij}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\Big) = h_{ik}h_{jl} - h_{il}h_{jk}\,,\\
\mathrm{Ric}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}\,h_{ij} - h_{il}g^{lk}h_{kj}\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = g^{ij}g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}^2 - |\mathrm{A}|^2\,.
\end{aligned}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}
\nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ks}X^l\,,\\
\nabla_i\nabla_j \omega_k - \nabla_j\nabla_i \omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ls}\omega_s\,.
\end{aligned}$$

The symmetry properties of the covariant derivative of A are given by the following Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} \tag{1.1.3}$$

which imply the following *Simons' identity* (see [109]),

$$\Delta h_{ij} = \nabla_i \nabla_j \mathrm{H} + \mathrm{H}\, h_{il} g^{ls} h_{sj} - |\mathrm{A}|^2 h_{ij}\,. \tag{1.1.4}$$

We will write $T * S$, following Hamilton [56], to denote a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair T, S with the metric g_{ij} and/or its inverse g^{ij}.
A very useful property of such $*$–product is that

$$|T * S| \leq C\,|T|\,|S|$$

where the constant C depends only on the algebraic "structure" of $T * S$.

Sometimes we will need the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, we will denote it by $\mathcal{H}^n$.

We advise the reader that in all the computations the constants could vary between different formulas and from a line to another.

1.2 First Variation of the Area Functional

Given an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$\mathrm{Area}(\varphi) = \int_M d\mu$$

where μ is the canonical measure associated to the metric g induced by the immersion.

In this section we are going to analyze the first variation of such functional which is nothing else than the volume of the hypersurface.

We consider a smooth one parameter family of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.
Defining the field $X = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ along M (as a submanifold of $\mathbb{R}^{n+1}$) we see that X is zero outside K, we call such field the *infinitesimal generator* of the variation φ_t.
We compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \varphi_t}{\partial x_i}\,\Big|\,\frac{\partial \varphi_t}{\partial x_j}\right\rangle\Big|_{t=0}\\
&= \left\langle \frac{\partial X}{\partial x_i}\,\Big|\,\frac{\partial \varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j}\,\Big|\,\frac{\partial \varphi}{\partial x_i}\right\rangle\\
&= \frac{\partial}{\partial x_i}\left\langle X\,\Big|\,\frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X\,\Big|\,\frac{\partial \varphi}{\partial x_i}\right\rangle - 2\left\langle X\,\Big|\,\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle\\
&= \frac{\partial}{\partial x_i}\left\langle X^M\,\Big|\,\frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X^M\,\Big|\,\frac{\partial \varphi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X^M\,\Big|\,\frac{\partial \varphi}{\partial x_k}\right\rangle - 2h_{ij}\langle X\,|\,\nu\rangle\,,
\end{aligned}$$

where X^M is the tangent component of the field X and we used the Gauss–Weingarten relations (1.1.1) in the last step.
Letting ω be the 1–form defined by $\omega(Y) = g(d\varphi^*(X^M), Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k - 2h_{ij}\langle X\,|\,\nu\rangle = \nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij}\langle X\,|\,\nu\rangle\,.$$

Hence, using the formula $\partial_t \det A(t) = \det A(t)\mathrm{Trace}[A^{-1}(t)\partial_t A(t)]$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\Big|_{t=0} &= \frac{\sqrt{\det(g_{ij})}\,g^{ij}\frac{\partial}{\partial t}g_{ij}\big|_{t=0}}{2}\\ &= \frac{\sqrt{\det(g_{ij})}\,g^{ij}\big(\nabla_i\omega_j+\nabla_j\omega_i-2h_{ij}\langle X\,|\,\nu\rangle\big)}{2}\\ &= \sqrt{\det(g_{ij})}\big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\,.\end{aligned}$$

If the Area of the immersion φ is finite, the same holds for all the maps φ_t, as they are compact deformations of φ. Assuming that the compact K is contained in a single coordinate chart, we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} &= \frac{\partial}{\partial t}\int_K d\mu_t\,\Big|_{t=0}\\ &= \frac{\partial}{\partial t}\int_K\sqrt{\det(g_{ij})}\,d\mathcal{L}^n\,\Big|_{t=0}\\ &= \int_K\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\Big|_{t=0}\,d\mathcal{L}^n\\ &= \int_K\big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\sqrt{\det(g_{ij})}\,d\mathcal{L}^n\\ &= \int_M\big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\,d\mu\\ &= -\int_M\mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\end{aligned}$$

where we used the fact that X is zero outside K and in the last step we applied the divergence theorem. Notice that all the integrals are well defined because we are actually integrating only on the compact set K.
In the case that K is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

Proposition 1.2.1. *The first variation of the* Area *functional depends only on the normal component of the infinitesimal generator* $X = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ *of the variation* φ_t*, precisely*

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} = -\int_M\mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\,.$$

Clearly, such dependence is linear.

Given any immersion $\varphi : M\to\mathbb{R}^{n+1}$ and any vector field X with compact support along M, we can always construct a variation with infinitesimal generator X as $\varphi_t(p) = \varphi(p)+tX(p)$. It is easy to see that for $|t|$ small the map φ_t is still a smooth immersion.
Hence, as the hypersurfaces which are critical points of the Area functional must satisfy

$$\int_M\mathrm{H}\langle X\,|\,\nu\rangle\,d\mu = 0$$

for every field X with compact support, they must have $\mathrm{H} = 0$ everywhere, that is, zero mean curvature (and conversely). This is the well known definition of the so called *minimal surfaces*.

As the quantity $-\mathrm{H}\nu$ can be interpreted as the *gradient* of the Area functional (be careful here, the measure μ is varying with the immersion, we are not computing the gradient with respect to some fixed L^2–structure on the space of immersions of M in $\mathbb{R}^{n+1}$), we can consider the motion of a hypersurface by minus this gradient, that is, the *mean curvature flow*. So, one looks for hypersurfaces moving with velocity $\mathrm{H}\nu$ at every point. This means choosing, among all the velocity functions with fixed $L^2(\mu)$–norm equal to $\left(\int_M \mathrm{H}^2\, d\mu\right)^{1/2}$, the one such that the Area of hypersurface decreases most rapidly.

This idea is quite natural and arises often in studying the dynamics of models of physical situations where the energy is given by the "Area" of the interfaces between the phases of a system. Moreover, as the Area functional is the simplest (in terms of derivatives of the parametrization) geometric functional, that is, invariant by isometries of $\mathbb{R}^{n+1}$ and diffeomorphisms of M, the motion by mean curvature is the simplest *variational* geometric flow for immersed hypersurfaces. Other geometric functionals (for instance, depending on the next simpler geometric invariant, the curvature) generally produce a first variation of order higher than two in the derivatives of the parametrization and a relative higher order PDE's system.
One can consider second order flows where the velocity of the motion is related to different functions of the curvature, like the Gauss flow of surfaces, for instance, where the velocity is given by $\mathrm{G}\nu$ (G is the Gauss curvature of M, that is, $\mathrm{G} = \det \mathrm{A}$) or more complicated flows, but these evolutions are usually not variational, they do not arise as "gradients" (in the above sense) of geometric functionals (see Section 1.6).

1.3 The Mean Curvature Flow

Definition 1.3.1. Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n–dimensional manifold. The mean curvature flow of φ_0 is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p,t) = \varphi_t(p)$ the map $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{1.3.1}$$

where $\mathrm{H}(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the unit normal of the hypersurface φ_t at the point $p \in M$.

Remark 1.3.2. Notice that even if the unit normal vector is defined up to a sign, the field $\mathrm{H}(p,t)\nu(p,t)$ is independent of such choice.

Using equation (1.1.2), this system can be rewritten in the appealing form

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi$$

but, despite its formal analogy with the heat equation, actually, it is a second order, *quasilinear* and *degenerate*, parabolic system, as the Laplacian is the one associated to the evolving hypersurfaces at time t,

$$\Delta\varphi(p,t) = \Delta_{g(p,t)}\varphi(p,t) = g^{ij}(p,t)\nabla_i^{g(p,t)}\nabla_j^{g(p,t)}\varphi(p,t)$$

and its coefficients as second order partial differential operator depend on the first derivatives of φ. Moreover, this operator is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms, see [49] for details.

Like the Area functional, the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{R}^{n+1}$. Moreover, if $\varphi(p,t)$ is a mean curvature flow and $\Psi : M \to M$ is a diffeomorphism, then the reparametrization $\widetilde{\varphi}(p,t) = \varphi(\Psi(p), t)$ is still a mean curvature flow. This last property can be reread as "the flow is invariant under reparametrization", suggesting that the important objects in the flow are actually the subsets $M_t = \varphi(M,t)$ of $\mathbb{R}^{n+1}$.

The problem also satisfies the following parabolic invariance under rescaling (consequence of the property $\mathrm{Area}(\lambda\varphi) = \lambda^n \mathrm{Area}(\varphi)$, for any n–dimensional immersion), if $\varphi(p,t)$ is a mean curvature flow of φ_0 and $\lambda > 0$, then $\widetilde{\varphi}(p,t) = \lambda\varphi(p, \lambda^{-2}t)$ is a mean curvature flow of the initial hypersurface $\lambda\varphi_0$.

During the flow the Area of the hypersurfaces (which is the natural energy of the problem) is nonincreasing, indeed, by the same computation for the first variation of such functional in the previous section, we have

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t) = \frac{\partial}{\partial t}\int_M d\mu_t = -\int_M \mathrm{H}^2\, d\mu_t\,.$$

This clearly implies the estimate

$$\int_0^{T_{\max}}\int_M \mathrm{H}^2\, d\mu_t \leq \mathrm{Area}(\varphi_0)$$

in the maximal time interval $[0, T_{\max})$ of smooth existence of the flow.

Exercise 1.3.3. By means of this last inequality, try to get an estimate from above for the maximal time of smooth existence $T_{\max}$ for closed curves in $\mathbb{R}^2$ and compact *surfaces* in $\mathbb{R}^3$.

Proposition 1.3.4 (Geometric Invariance under Tangential Perturbations)**.** *If a smooth family of immersions $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfies the system of PDE's*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) + X(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases}$$

where X is a time dependent smooth vector field along M such that $X(p,t)$ belongs to $d\varphi_t(T_pM)$ for every $p \in M$ and every time $t \in [0,T)$, then, locally around any point in space and time, there exists a family of reparametrizations (smoothly time dependent) of the maps φ_t which satisfies system (1.3.1).
If the hypersurface M is compact, one can actually find uniquely a family of global *reparametrizations of the maps φ_t as above for every $t \geq 0$, leaving the initial immersion φ_0 unmodified and satisfying system* (1.3.1).
Conversely, if a smooth family of moving hypersurfaces $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ can be globally reparametrized for $t \geq 0$ in order that it moves by mean curvature, then the map φ has to satisfy the system above for some (uniquely determined) time dependent vector field X with $X(p,t) \in d\varphi_t(T_pM)$, for every $p \in M$ and every time $t \in [0,T)$.

Proof. First we assume that M is compact, we will produce a smooth global parametrization of the evolving sets in order to check Definition 1.3.1.
By the tangency hypothesis, the time dependent vector field on M given by

$$Y(q,t) = -d\varphi_t^*(q)(X(q,t))$$

is globally well defined and smooth.
Let $\Psi : M \times [0,T) \to M$ be a smooth family of diffeomorphisms of M with $\Psi(p,0) = p$ for every $p \in M$ and

$$\frac{\partial}{\partial t}\Psi(p,t) = Y(\Psi(p,t),t)\,, \tag{1.3.2}$$

for every time $t \in [0,T)$.
This family exists, is unique and smooth, by the existence and uniqueness theorem for ODE's on the compact manifold M.

Considering the reparametrizations $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$, one has

$$\begin{aligned}\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) =& \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\ =& \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))(Y(\Psi(p,t),t))\\ =& \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t)\\ &- d\varphi_t(\Psi(p,t))(d\varphi_t^*(\Psi(p,t))(X(\Psi(p,t),t)))\\ =& \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\\ =& \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\,.\end{aligned}$$

Hence, $\widetilde{\varphi}$ satisfies system (1.3.1) and $\widetilde{\varphi}_0 = \varphi_0$.
Conversely, this computation also shows that if $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ satisfies system (1.3.1), the family of diffeomorphisms $\Psi : M \times [0,T) \to M$ must solve equation (1.3.2), hence, it is unique if we assume $\Psi(\,\cdot\,,0) = \mathrm{Id}_M$ in order that the map φ_0 is unmodified.
In the noncompact case, we have to work locally in space and time, solving the above system of ODE's in some positive interval of time in an open subset $\Omega \subset M$ with compact closure, then obtaining a solution of system (1.3.1) in a possibly smaller open subset of Ω and some interval of time.

Assume now that the reparametrized map $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ is a mean curvature flow. Differentiating, we get

$$\begin{aligned}\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) =& \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\ =& \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\\ =& \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\end{aligned}$$

that is,

$$\frac{\partial\varphi}{\partial t}(q,t) = \mathrm{H}(q,t)\nu(q,t) - d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)\,,$$

for every $q \in M$ and $t \in [0,T)$. Then, the last statement of the proposition follows by setting $X(q,t) = -d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)$. □

Corollary 1.3.5. *If a smooth family of hypersurfaces $\varphi_t = \varphi(\,\cdot\,,t)$ satisfies $\langle\partial_t\varphi\,|\,\nu\rangle = \mathrm{H}$, then it can be everywhere locally reparametrized to a mean curvature flow. If M is compact, this can be done uniquely by global reparametrizations, without modifying φ_0.*

Remark 1.3.6. A short way to state the previous proposition and corollary is to say that the tangential component of the velocity of the points of the hypersurface, does not affect the global "shape" during the motion.
This is particularly meaningful in the case that system (1.3.1) has a unique solution, for instance when M is compact, as we will see in Theorem 1.5.1 in the next section.

By this invariance property one is led to speak of mean curvature flow of hypersurfaces considering them as subsets of $\mathbb{R}^{n+1}$ and forgetting their parametrizations. This is clear in the case of embedded hypersurfaces, where the identification of $(M, g(t))$ with the images of the embeddings $\varphi_t(M)$ is immediate, but it also works for nonembedded hypersurfaces as every immersion is locally an embedding.
We give then a more geometric, alternative definition of the mean curvature flow. In the sequel it will be clear by the context which one we are using.

Definition 1.3.7. We still say that a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, for $t \in [0,T)$, is a mean curvature flow if locally at every point, in space and time, there exists a family of reparametrizations which satisfies system (1.3.1).

Proposition 1.3.4 expresses the substantial equivalence between this definition (Eulerian point of view) and Definition 1.3.1 (Lagrangian point of view).

Exercise 1.3.8 (Motion of Graphs)**.** Show that if the smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, are locally graphs on some open subset Ω of the hyperplane $\langle e_1, \dots, e_n \rangle \subset \mathbb{R}^{n+1}$, that is, we have a smooth function $f : \Omega \times [0, T) \to \mathbb{R}$, such that

$$\varphi(p, t) = (x_1(p), \dots, x_n(p), f(x_1(p), \dots, x_n(p), t)) ,$$

there holds

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = \sqrt{1 + |\nabla f|^2}\, \mathrm{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) .$$

On the other hand, if we have a function f satisfying the above parabolic equation then its graph is a hypersurface moving by mean curvature (according to Definition 1.3.7).

Exercise 1.3.9 (Motion of Level Sets)**.** Assume that for every time $t \in [0, T)$ the image $\varphi_t(M)$ of the smooth, embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, is the *zero set* of $f_t = f(\,\cdot\,, t)$, where $f : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ is a smooth function and zero is a regular value of f_t for every $t \in [0, T)$. Then at all the points $x \in \mathbb{R}^{n+1}$ and times $t \in [0, T)$ such that $f(x, t) = 0$ there holds

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^2} = |\nabla f|\, \mathrm{div} \left(\frac{\nabla f}{|\nabla f|} \right) .$$

Conversely, if we have a smooth function f satisfying the above parabolic equation, *every regular* level set of $f(\,\cdot\,, t)$ is a hypersurface moving by mean curvature (according to Definition 1.3.7).

Exercise 1.3.10 (Distance Functions)**.** Compute the evolution equation satisfied by the signed distance function $d_{M_t} : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ at the points $x \in M_t = \varphi_t(M)$, if the compact and embedded, smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ move by mean curvature.
The signed distance function is the function which coincides with the distance in the region "outside" a hypersurface and with minus the distance in the "inside" region (show that it is smooth in a tubular neighborhood of the hypersurface).

Exercise 1.3.11 (Brakke's Definition [21])**.** Show that a smooth family of compact and embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ satisfies

$$\frac{d}{dt} \int_{\varphi_t(M)} f \, d\mathcal{H}^n \leq \int_{\varphi_t(M)} \left(\mathrm{H} \langle \nabla f \,|\, \nu \rangle - \mathrm{H}^2 f \right) \, d\mathcal{H}^n ,$$

for every *positive* function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, if and only if the hypersurfaces are moving by mean curvature flow (according to Definition 1.3.7).
In the formula $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

1.4 Examples

Spheres and cylinders are the easiest and actually some of the few nontrivial explicitly computable examples of mean curvature flows (minimal surfaces are trivial examples as they are not moving at all, satisfying $\mathrm{H} = 0$).

Let us consider a sphere of radius R which, by the translation invariance of the flow, we can assume to be centered at the origin of $\mathbb{R}^{n+1}$. A right guess is that at every time the hypersurface remains a sphere and the mean curvature flow simply changes its radius $R(t)$, this is actually true by the uniqueness theorem in the next section. As the mean curvature is everywhere equal to n/R and since we chose the inner pointing unit normal, the evolution equation for the radius of the sphere is simply $R'(t) = -n/R(t)$ with $R(0) = R$. Indeed, if we set $M = \mathbb{S}^n$ and $\varphi(p,t) = R(t)\varphi_0(p)$, being φ_0 the standard immersion of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we have

$$R'(t)\varphi_0(p) = \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) = -n\varphi_0(p)/R(t)\,,$$

which is an ODE that can be easily integrated to get $R(t) = \sqrt{R^2 - 2nt}$.
At time $T_{\max} = R^2/(2n)$ the sphere shrinks to a point so the flow becomes singular, this is the maximal time of existence. We can then write the evolution of the radius also as $R(t) = \sqrt{2n(T_{\max} - t)}$.
During the flow the norm of the second fundamental form evolves as

$$|\mathrm{A}(t)| = \sqrt{n}/R(t) = \frac{1}{\sqrt{2(T_{\max} - t)}}\,.$$

Other examples are given by the cylinders $\mathbb{S}^m(R) \times \mathbb{R}^{n-m}$. In general, we can see that if $\varphi : M \times [0,T) \to \mathbb{R}^{m+1}$ is a mean curvature flow of an m–dimensional hypersurface M of $\mathbb{R}^{m+1}$, then the map $\widetilde{\varphi} : (M \times \mathbb{R}^{n-m}) \times [0,T) \to \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} = \mathbb{R}^{n+1}$, defined by $\widetilde{\varphi}(p,s,t) = (\varphi(p,t), s)$, is a mean curvature flow of the immersion of the product manifold $M \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$.
Then, by the above discussion, these cylinders evolve homothetically as $\mathbb{S}^m(R(t)) \times \mathbb{R}^{n-m}$, with $R(t) = \sqrt{R^2 - 2mt}$ and collapse to the subspace $\{0\} \times \mathbb{R}^{n-m}$ at time $T_{\max} = R^2/(2m)$. Again, the norm of the second fundamental form satisfies $|\mathrm{A}(t)| = \frac{1}{\sqrt{2(T_{\max}-t)}}$.
As these cylinder are noncompact, it must be remarked here that it is needed here a uniqueness theorem also for noncompact hypersurfaces to conclude that their evolution is actually only the one described above. In this case it actually holds, see Remark 1.5.4.

Spheres and cylinders are special examples of *homothetically shrinking* flows, that is, hypersurfaces that simply move by contraction during the evolution by mean curvature.

Proposition 1.4.1. *If an initial hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$ satisfies*

$$\mathrm{H}(p) + \lambda\langle\varphi_0(p) - x_0 \,|\, \nu_0(p)\rangle = 0$$

at every point $p \in M$ for some constant $\lambda > 0$ and $x_0 \in \mathbb{R}^{n+1}$, then it generates a homothetically shrinking mean curvature flow (according to Definition 1.3.7) around the point $x_0 \in \mathbb{R}^{n+1}$, described by

$$\varphi(p,t) = x_0 + \sqrt{1 - 2\lambda t}\,(\varphi_0(p) - x_0)$$

in the time interval $[0, 1/2\lambda)$.
Conversely, if $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a homothetically shrinking mean curvature flow (according to Definition 1.3.7) around some point $x_0 \in \mathbb{R}^{n+1}$ in a maximal time interval, then either H *is identically zero or*

$$\mathrm{H}(p,t) + \frac{\langle\varphi(p,t) - x_0 \,|\, \nu(p,t)\rangle}{2(T-t)} = 0\,,$$

for every point $p \in M$ and time $t \in [0,T)$.

Proof. If the condition is satisfied, we consider the homothetically shrinking flow

$$\varphi(p,t) = x_0 + \sqrt{1 - 2\lambda t}\,(\varphi_0(p) - x_0)$$

and we see that

$$\langle\partial_t\varphi(p,t) \,|\, \nu(p,t)\rangle = -\frac{\lambda\langle\varphi_0(p) - x_0 \,|\, \nu(p,t)\rangle}{\sqrt{1-2\lambda t}} = \frac{\mathrm{H}(p,0)}{\sqrt{1-2\lambda t}} = \mathrm{H}(p,t)\,,$$

as $\nu(p,t) = \nu_0(p)$. Hence, by Corollary 1.3.5, this is a mean curvature flow of the initial hypersurface φ_0, according to Definition 1.3.7.

Conversely, if the homothetically shrinking evolution $\varphi(p,t)=x_0+f(t)(\varphi_0(p)-x_0)$ is a mean curvature flow, for some positive smooth function $f:[0,T)\to\mathbb{R}$ with $f(0)=1$, $\lim_{t\to T}f(t)=0$ and $f'(t)\leq 0$, by Corollary 1.3.5 we have $\langle\partial_t\varphi\,|\,\nu\rangle=\mathrm{H}$, hence

$$\begin{aligned}\mathrm{H}(p,0)&=f(t)\mathrm{H}(p,t)\\&=f(t)\langle\partial_t\varphi(p,t)\,|\,\nu(p,t)\rangle\\&=f(t)f'(t)\langle\varphi_0(p)-x_0\,|\,\nu(p,t)\rangle\\&=f'(t)\langle\varphi(p,t)-x_0\,|\,\nu(p,t)\rangle\,.\end{aligned}$$

If $\mathrm{H}\neq 0$ at some point, as $\nu(p,t)=\nu_0(p)$ we have that $f(t)f'(t)$ is equal to some constant C for every $t\in[0,T)$, combining the first and the third line of the above formula. Hence, $f(t)=\sqrt{2Ct+1}$ as $f(0)=1$ and since $\lim_{t\to T}f(t)=0$, we conclude $f(t)=\sqrt{1-t/T}$. The thesis then follows from the first and last line of the formula. □

We underline that again $|\mathrm{A}(t)|\sim\frac{1}{\sqrt{2(T_{\max}-t)}}$.
Up to a rigid motion and rescaling, solving the above structural equation is equivalent to solving $\mathrm{H}+\langle\varphi_0\,|\,\nu_0\rangle=0$. The hypersurfaces satisfying this equation are often called *shrinkers*, they generate a homothetically shrinking mean curvature flow (according to Definition 1.3.7) around the origin of $\mathbb{R}^{n+1}$, in the time interval $[0,1/2)$.
In the special case of curves in $\mathbb{R}^2$, supposing γ parametrized by its arclength s, the unit tangent vector is given by $\tau=\gamma_s$ and the unit normal by $\nu=\mathrm{R}\gamma_s$, where $\mathrm{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the counterclockwise rotation of $\pi/2$. Then, such equation becomes

$$k+\langle\gamma\,|\,\nu\rangle=0\qquad\text{that is,}\qquad\gamma_{ss}=k\nu=-\langle\gamma\,|\,\nu\rangle\nu=-\langle\gamma\,|\,\mathrm{R}\gamma_s\rangle\mathrm{R}\gamma_s\,,$$

hence one can find the homothetically shrinking curves by integrating this ODE.
The only embedded solutions are the circle $\mathbb{S}^1$ and the lines through the origin of $\mathbb{R}^2$ (see Appendix E), but there are also several other nonembedded closed curves found by Abresch and Langer [1] that classified all the possible solutions, see also the work of Epstein and Weinstein [41].

Finding homothetically shrinking hypersurfaces when the dimension is higher than one is difficult (see the discussion in [81]). It is known that besides the "standard" examples given by the hyperplanes through the origin, the spheres and the cylinders, there exists a homothetically shrinking, embedded torus in $\mathbb{R}^3$, found by Angenent [17] (it seems that Grayson was the first to suggest its existence, see [68]). Moreover, there is numerical evidence that higher genus surfaces in $\mathbb{R}^3$ could also exist, see Chopp [25] and Angenent, Chopp and Ilmanen [18] (see also [81]). Finally, a tentative strategy to produce new examples is being carried on in some recent papers by Nguyen [98, 97, 99]).

One can also look for homothetically *expanding* hypersurfaces (around the origin of $\mathbb{R}^{n+1}$), which are characterized by the same equation $\mathrm{H}+\lambda\langle\varphi_0\,|\,\nu_0\rangle=0$ but with a negative constant λ. These cannot be compact, as one can see easily by looking at their point of maximum distance from the origin of $\mathbb{R}^{n+1}$.
As an example, every angle less than π in $\mathbb{R}^2$ contains a convex, unbounded, homothetically expanding curve under the curvature flow, asymptotic to the edges of the angle (see also the discussion in [21, Appendix C]). A classification of graph solutions of the above equation with at most linear growth can be found in [117].

Another notable family of hypersurfaces moving by mean curvature are the ones generating *translating* flows, these are hypersurfaces that during the motion do not change their shape but simply move in a fixed direction with constant velocity.

Proposition 1.4.2. *If an initial hypersurface $\varphi_0:M\to\mathbb{R}^{n+1}$ satisfies $\mathrm{H}(p)=\langle v\,|\,\nu_0(p)\rangle$ at every point $p\in M$ for some constant vector $v\in\mathbb{R}^{n+1}$, then it generates a translating mean curvature flow (according to Definition 1.3.7) with constant velocity v.*
Conversely, if $\varphi:M\times[0,T)\to\mathbb{R}^{n+1}$ is a translating mean curvature flow (according to Definition 1.3.7) then there exists a vector $v\in\mathbb{R}^{n+1}$ (which is the velocity of the motion) such that $\mathrm{H}(p,t)=\langle v\,|\,\nu(p,t)\rangle$ for every point $p\in M$ and $t\in[0,T)$.

Proof. If the condition is satisfied, we consider the translating flow $\varphi(p,t) = \varphi_0(p) + tv$ and we see that

$$\langle \partial_t \varphi(p,t) \,|\, \nu(p,t) \rangle = \langle v \,|\, \nu(p,t) \rangle = \langle v \,|\, \nu_0(p) \rangle = \mathrm{H}(p,0) = \mathrm{H}(p,t)\,,$$

as $\nu(p,t) = \nu_0(p)$. Hence, by Corollary 1.3.5, this is a mean curvature flow of the initial hypersurface φ_0, according to Definition 1.3.7.

Conversely, if the translating flow $\varphi(p,t) = \varphi_0(p) + w(t)$ is a mean curvature flow, for some smooth, time dependent vector $w : [0,T) \to \mathbb{R}^{n+1}$ with $w(0) = 0$, by Corollary 1.3.5 we have $\langle \partial_t \varphi \,|\, \nu \rangle = \mathrm{H}$, hence

$$\langle \partial_t \varphi(p,t) \,|\, \nu(p,t) \rangle = \langle w'(t) \,|\, \nu(p,t) \rangle = \mathrm{H}(p,t) = \mathrm{H}(p,0)\,.$$

Suppose that varying p in M, the image of the unit normal is a subset of $\mathbb{R}^{n+1}$ whose span is the whole $\mathbb{R}^{n+1}$, then, as $\nu(p,t) = \nu_0(p)$, if we differentiate in time the equality $\langle w'(t) \,|\, \nu_0(p) \rangle = \mathrm{H}(p,0)$, we get $\langle w''(t) \,|\, \nu_0(p) \rangle = 0$ which implies that $w''(t) = 0$ for every t and $w'(t)$ constant. Then, letting $w' = v$, we have the thesis.

In the case that the span of the image of the unit normal is not the whole $\mathbb{R}^{n+1}$, all the tangent spaces T_pM to φ_0 have a common nontrivial vector subspace $L \subset \mathbb{R}^{n+1}$. Decomposing $w(t) = l(t) + z(t)$ with $l(t) \in L$ and $z(t) \in L^\perp$ we have $l(0) = z(0) = 0$ and $z''(t) = 0$ by the above argument, as $l'(t), l''(t) \in L$ and $z'(t), z''(t) \in L^\perp$. Hence, $z'(t)$ is constant and

$$\mathrm{H}(p,t) = \langle w'(t) \,|\, \nu(p,t) \rangle = \langle z'(t) \,|\, \nu(p,t) \rangle = \langle v \,|\, \nu(p,t) \rangle$$

where we set $v = z'$.
By Proposition 1.3.4 (see also Remark 1.3.6) the translating flow $\widetilde{\varphi}(p,t) = \varphi_0(p) + tv$ coincides (as sets) with the flow φ as $\partial_t \widetilde{\varphi} = v$, $\partial_t \varphi = w'(t) = l'(t) + v$ and $l'(t) \in L \subset T_pM$ for every $p \in M$ and $t \in [0,T)$. $\square$

In the special case of curves in $\mathbb{R}^2$, if γ is parametrized by its arclength s and the unit normal is $\nu = \mathrm{R}\gamma_s$ as before, the above equation becomes

$$k = \langle v \,|\, \nu \rangle \qquad \text{that is,} \qquad \gamma_{ss} = k\nu = \langle v \,|\, \nu \rangle \nu = \langle v \,|\, \mathrm{R}\gamma_s \rangle \mathrm{R}\gamma_s\,,$$

where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of $\pi/2$. By integrating such ODE, one can see that the only possible translating curve is given (up to homotheties and rigid motions) by the graph of the function $x = -\log\cos y$ in the interval $(-\pi/2, \pi/2)$, which was called the *grim reaper* by Grayson [52].

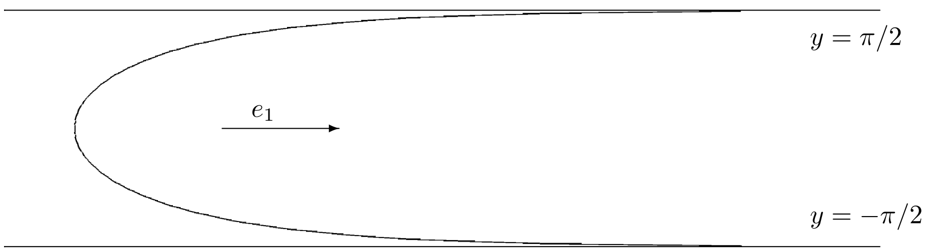

Figure 1.1: The *grim reaper* moving with constant velocity e_1.

In higher dimension, for every fixed vector $v \in \mathbb{R}^{n+1}$ there is a unique rotationally symmetric, strictly convex hypersurface (which is actually an entire graph) moving by translation under the mean curvature flow. Indeed, looking for a convex graph over a domain in $\mathbb{R}^n$ (identified with the hyperplane $\{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$), translating in the e_{n+1} direction with unit speed, one has to find a convex function $f : \Omega \to \mathbb{R}$ such that

$$\Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = 1$$

and $f(0) = \nabla f(0) = 0$, where $\mathrm{Hess}f$ is the Hessian of f.
Imposing rotational symmetry around the origin $f(x) = f(\rho)$ with $\rho = |x|$, this problem becomes the following ODE

$$f_{\rho\rho} + \frac{(n-1)f_\rho}{\rho} - \frac{f_{\rho\rho}f_\rho^2}{1+f_\rho^2} = 1\,,$$

that is,

$$f_{\rho\rho} = (1+f_\rho^2)\left(1 - \frac{(n-1)f_\rho}{\rho}\right) \tag{1.4.1}$$

with $\lim_{\rho\to 0} f(\rho) = \lim_{\rho\to 0} f'(\rho) = 0$ for a convex function $f : \mathbb{R}^+ \to \mathbb{R}$.
When $n = 1$ the solution of this ODE gives the *grim reaper*, when $n > 1$ there is only one solution, defined on all $\mathbb{R}^+$ and growing quadratically at infinity. This solution provides the only rotationally symmetric, convex, translating hypersurface moving by mean curvature, up to homotheties and rigid motions.

Exercise 1.4.3. Show the claimed properties of the solution of such ODE.

In the paper by Wang [122] it is proved that in dimension two every convex and translating flow must be either the product of a grim reaper with $\mathbb{R}$ or a rotationally symmetric strictly convex entire graph and that in every dimension larger than two there exist examples which are not rotationally symmetric and also non entire solutions of the above PDE (actually, convex solutions defined in strips of $\mathbb{R}^n$). See also [124, page 536] and [55].
Recently, Nguyen [98, 100] exhibited some new nonconvex, embedded examples of translating hypersurfaces, with a trident–like shape at "large scales".

Finally, there are also rotating (or rotating and dilating/contracting) flows, see [77, 82] and [27, Section 2.2], like the unbounded spiral in the plane called the *Yin–Yang* curve that rotates during its motion by curvature, depicted in the paper [3] by Altschuler.

We have seen that the homothetically shrinking hypersurfaces cannot "live" forever, at some maximal time $T_{\max} > 0$ the map φ becomes singular. This is a common fact to any compact initial hypersurface, as we will see in Corollary 2.2.5.
There are two possible reasons why this happens: the first is analytic, the function φ could stop being smooth, usually because some derivatives of φ are not bounded as $t \to T_{\max}$, the second reason is geometric, when the map φ stops being an immersion, that is, $d\varphi_t$ becomes singular (not one–to–one) at some point of M and time $t = T_{\max}$. We will see that in both cases the curvature of the evolving hypersurface has to become unbounded, that is, if the second fundamental form A stays uniformly bounded till $T_{\max}$ then this latter cannot be the maximal time of existence of a smooth flow (Proposition 2.4.6).

In all the above self–similar examples, either the flow is smoothly defined for every positive time or at some finite time the hypersurface instantly completely vanishes. This is quite a special behavior, indeed, in general the singularities develop only in some regions of the evolving hypersurface. An example of a more generic and "concrete" singularity is a nonembedded cardioid curve in the plane with a small loop: at some time the small loop has shrunk while the rest of the curve has remained smooth and a cusp has developed (see [14, 15, 16]).
In this example the initial curve was not embedded (we will see in the sequel why this was necessary), another example, this time embedded, of what can happen at the singular time is the *dumbbell* surface [53], or *standard neckpinch*.
Consider a long thin cylinder (the neck) in $\mathbb{R}^3$ smoothly connecting symmetrically two large spheres at its ends (it is possible to construct an example of such a surface also with $\mathrm{H} > 0$), then, during the flow one can guess that the cylinder, which has a large positive mean curvature, shrinks faster than the two big spheres at its ends which share instead a small curvature, having a large radius. Then, the cylinder collapses at some time and the hypersurface tends to the symmetric union of two "water drops" joined at the vertices of their cusps.
The existence of surfaces with this behavior under the mean curvature flow was first proved rigorously by Grayson [53]. Another similar example was worked out by Huisken in [68]. The

following simple argument can be found in [17]: consider the Angenent's homothetically shrinking torus, mentioned above, surrounding the "cylindrical" part of a symmetric dumbbell surface and a couple of spheres "inside" the two side balls, in a way that neither the torus nor the two spheres touch the dumbbell. We will see in Theorem 2.2.1 that then the four surfaces cannot touch each other during all their flows, as they have no intersections at the initial time. If the dumbbell is chosen in such a way that the torus (which is shrinking homothetically to a point) has an extinction time T smaller than the one of the two "inside" spheres, the dumbbell must develop some kind of singularity before T since it is "squeezed" by the torus, but it cannot vanish as it has still to "contain" the two spheres.

Our last example is the standard torus in $\mathbb{R}^3$ obtained by rotating around the z axis a very small circle with a center far enough from such axis (if the circle is small enough with respect to its distance from the z axis, this torus has $\mathrm{H} > 0$). During the evolution by mean curvature, the circle tends to shrink before the whole torus has time to collapse, hence, by the rotational symmetry which is maintained during the flow, one expects that at the maximal time the torus develops a circle of singularities around the z axis (this will be discussed in detail in Section 4.6). This suggests that in general the limit shape of an evolving hypersurface at a singular time can be quite wild (some results on the "size" of the singular sets have been obtained by White [123]).

These last examples motivated the large research in literature about suitable weak solutions of the mean curvature flow, in order to define a generalized evolution even after a singular time. In a physical model, indeed, the interface could continue the evolution even after such time, possibly in a nonsmooth way. In the dumbbell situation above, for instance, we intuitively expect that the surface splits in two parts, each of them moving independently after the splitting. Also for topological applications, in the same spirit of Hamilton's program for the Ricci flow, it is important to be able to continue the flow after any singular time until the hypersurface (or all the parts in which it separates) converges to some known limit.
Weak solutions of the mean curvature flow have been introduced by many authors in different ways; among the others, we recall the definition by Brakke [21], based on geometric measure theory and the ones by Chen, Giga, Goto [24] and Evans, Spruck [43] based on the level sets formulation and the theory of viscosity solutions.

1.5 Short Time Existence of the Flow

Theorem 1.5.1. *For any initial, smooth and compact hypersurface of $\mathbb{R}^{n+1}$ given by an immersion $\varphi_0 : M \to \mathbb{R}^{n+1}$, there exists a unique, smooth solution of system* (1.3.1) *in some positive time interval. Moreover, the solution continuously depends in C^∞ on the initial immersion φ_0.*

Remark 1.5.2. In literature this result was proved in several ways:

- Gage and Hamilton used the Nash–Moser inverse function theorem, actually a very strong tool for the existence of solutions of parabolic systems of PDE's (we recall that we are dealing with a *degenerate quasilinear* system), see [49]. To the author's knowledge, this is the first published proof of smooth existence for short time, moreover it works in general for the motion by mean curvature of a compact submanifold of any codimension immersed in a Riemannian manifold.

- Evans and Spruck in [43, 44] showed the existence of a smooth solution of the PDE satisfied by the distance function from an embedded hypersurface moving by mean curvature (see also [90]).

- Huisken and Polden in [73] (or in the Ph.D. Thesis of Polden [104]) reduced the degenerate parabolic system to a *nondegenerate* parabolic *equation*, representing the evolving hypersurfaces as graphs over the initial one in a tubular neighborhood of this latter (this line is mentioned also in [17] and [39]).
  This is the proof that we are going to show below and which is the most natural one in the

"classical" approach to mean curvature flow. Moreover it is quite elastic to be adapted to a very general class of flows of hypersurfaces and to be generalized to any codimension and any Riemannian ambient space (see [40]).

Another possibility is to use some version of the so called *DeTurck's trick* (see [34]), coupling the mean curvature flow with another flow, in order to eliminate the diffeomorphism invariance of the problem (see [129] and [23]).

Moreover, there are also many existence proofs of generalized evolutions by mean curvature (after introducing weak definitions of hypersurfaces) of nonregular or possibly "wild" subsets of $\mathbb{R}^{n+1}$, some even allow these latter to be merely closed sets in the Euclidean space. It should be said that, considering nonsmooth subsets in these generalized definitions of the flow, the uniqueness fails in several situations.
We mention some of these approaches:

- The use of theory of viscosity solutions to study the PDE satisfied by the function f in the formulation of the motion via level sets mentioned in Exercise 1.3.9, exploited by Evans and Spruck in [43, 44] and Chen, Giga and Goto in [24].
- The study by Soner [112] of the evolution equation satisfied by the signed distance function from an embedded hypersurface moving by mean curvature (see Exercise 1.3.10) by means of barrier comparison arguments and Perron's method.
- The varifold approach of Brakke using geometric measure theory, see [21] (a hint of Brakke's weak definition of mean curvature flow is given in Exercise 1.3.11).
- Almgren, Taylor and Wang discretization–minimization procedure in [2].
- Ilmanen's approximation in [78, 79].

Remark 1.5.3. One can show that the mean curvature flow shares a kind of the usual regularizing property of parabolic equations, for instance, any C^2 initial hypersurface becomes analytic at every positive time, in the sense that it is not the map φ_t which becomes analytic, but the image hypersurface $\varphi_t(M) \subset \mathbb{R}^{n+1}$, that is, it admits an analytic reparametrization.
Moreover, with the right definition, one can let evolve a hypersurface with corners or other singularities and these latter immediately vanish, see for instance [14, 16] and [39, 121].

Proof. We follow Huisken and Polden in [73].
Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact n–dimensional manifold. For the moment we assume that this hypersurface is embedded, hence the inner pointing unit normal vector field ν_0 is globally defined and smooth.
We look for a smooth solution $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ of the parabolic problem

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases}$$

for some $T > 0$.
Since we are interested in a solution for short time, we can forget about the immersion condition ($d\varphi_t$ nonsingular) as it will follow automatically by the smoothness of the solution and by the fact that φ_0 is a compact immersion, when t is close to zero.
Keeping in mind Proposition 1.3.4 and Corollary 1.3.5, if we find a smooth solution $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ of the problem

$$\begin{cases} \left\langle \frac{\partial}{\partial t}\varphi(p,t) \,\middle|\, \nu(p,t) \right\rangle = \mathrm{H}(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{1.5.1}$$

then we are done.
We consider the regular tubular neighborhood $\Omega = \{x \in \mathbb{R}^{n+1} \,|\, d(x, \varphi_0(M)) < \varepsilon\}$, which exists

for $\varepsilon > 0$ small enough. By regular we mean that the map $\Psi : M \times (-\varepsilon, \varepsilon) \to \Omega$ defined as $\Psi(p, s) = \varphi_0(p) + s\nu_0(p)$ is a diffeomorphism.
Any small C^∞ deformation of $\varphi_0(M)$ inside Ω can be represented as the graph of a "height" function f over $\varphi_0(M)$ (see later for the details) and conversely, to any function $f : M \to (-\varepsilon, \varepsilon)$ we can associate the hypersurface $M_f \subset \Omega$ given by $\varphi(p) = \varphi_0(p) + f(p)\nu_0(p)$. We want to compute now the equation for a smooth function f, time dependent, in order that such map φ satisfies system (1.5.1).
Obviously, as $f(\,\cdot\,, 0)$ gives the hypersurface φ_0, we have $f(p, 0) = 0$ for every $p \in M$.
First we compute the metric and the normal of the perturbed hypersurfaces, we set $f_i = \partial_i f$ then,

$$
\begin{aligned}
g_{ij}(p,t) &= \left\langle \frac{\partial \varphi(p,t)}{\partial x_i} \,\middle|\, \frac{\partial \varphi(p,t)}{\partial x_j} \right\rangle \\
&= \left\langle \frac{\partial \varphi_0}{\partial x_i} + f_i \nu_0 - f h_i^k(0) \frac{\partial \varphi_0}{\partial x_k} \,\middle|\, \frac{\partial \varphi_0}{\partial x_j} + f_j \nu_0 - f h_j^l(0) \frac{\partial \varphi_0}{\partial x_l} \right\rangle \\
&= \left\langle \frac{\partial \varphi_0}{\partial x_i} - f h_i^k(0) \frac{\partial \varphi_0}{\partial x_k} \,\middle|\, \frac{\partial \varphi_0}{\partial x_j} - f h_j^l(0) \frac{\partial \varphi_0}{\partial x_l} \right\rangle + f_i f_j \\
&= g_{ij}(p,0) - 2f(p,t)h_{ij}(p,0) + f^2(p,t)h_{ik}(p,0)g^{kl}(p,0)h_{lj}(p,0) + f_i(p,t)f_j(p,t)
\end{aligned}
$$

where we used Gauss–Weingarten equations (1.1.1).
The vectors

$$
\frac{\partial \varphi(p,t)}{\partial x_i} = \frac{\partial \varphi_0(p)}{\partial x_i} + f_i(p,t)\nu_0(p) - f(p,t)h_i^k(p,0)\frac{\partial \varphi_0(p)}{\partial x_k}
$$

generate the tangent space, hence the normal $\nu(p,t)$ is given by

$$
\begin{aligned}
\nu(p,t) &= \frac{\nu_0(p) - \left\langle \nu_0(p) \,\middle|\, \frac{\partial \varphi(p,t)}{\partial x_i} \right\rangle g^{ij}(p,t) \frac{\partial \varphi(p,t)}{\partial x_j}}{\left| \nu_0(p) - \left\langle \nu_0(p) \,\middle|\, \frac{\partial \varphi(p,t)}{\partial x_i} \right\rangle g^{ij}(p,t) \frac{\partial \varphi(p,t)}{\partial x_j} \right|} \\
&= \frac{\nu_0(p) - f_i(p,t) g^{ij}(p,t) \frac{\partial \varphi(p,t)}{\partial x_j}}{\left| \nu_0(p) - f_i(p,t) g^{ij}(p,t) \frac{\partial \varphi(p,t)}{\partial x_j} \right|}\,.
\end{aligned}
$$

Notice that the normal, the metric and thus its inverse depend only on first space derivatives of the function f. Moreover, as $f(p, 0) = \nabla f(p, 0) = 0$, everything is smooth and since M is compact, when t is small the denominator of the above expression for the normal is uniformly bounded below away from zero (actually it is close to one).
Then, we find out the second fundamental form,

$$
\begin{aligned}
h_{ij}(p,t) = \Big\langle \nu(p,t) \,\Big|\, & f_{ij}(p,t)\nu_0(p) + \frac{\partial^2 \varphi_0(p)}{\partial x_i \partial x_j} - f_i(p,t) h_j^k(p,0) \frac{\partial \varphi_0(p)}{\partial x_k} \\
& - f_j(p,t) h_i^l(p,0) \frac{\partial \varphi_0(p)}{\partial x_l} + f(p,t) \frac{\partial^2 \nu_0(p)}{\partial x_i \partial x_j} \Big\rangle \\
= \langle \nu(p,t) \,|\, & f_{ij}(p,t)\nu_0(p) \rangle + P_{ij}(p, f(p,t), \nabla f(p,t))
\end{aligned}
$$

where P_{ij} is a smooth form when f and ∇f are small, hence for t small.
Computing in normal coordinates around $p \in M$ with respect to the metric $g(t)$, the mean curvature is then given by

$$
\begin{aligned}
\mathrm{H}(p,t) &= g^{ij}(p,t) h_{ij}(p,t) \\
&= \langle \nu(p,t) \,|\, \nu_0(p) \rangle f_{ij}(p,t) g^{ij}(p,t) + P_{ij}(p, f(p,t), \nabla f(p,t)) g^{ij}(p,t) \\
&= \langle \nu(p,t) \,|\, \nu_0(p) \rangle \Delta_{g(t)} f(p,t) + P(p, f(p,t), \nabla f(p,t))\,,
\end{aligned}
$$

where P is a smooth function for f and ∇f small.
We are finally ready to write down the condition $\langle \partial_t \varphi \,|\, \nu \rangle = \mathrm{H}$ in terms of the function f,

$$\begin{aligned}\frac{\partial f(p,t)}{\partial t}\langle \nu_0(p)\,|\,\nu(p,t)\rangle &= \langle \partial_t\varphi(p,t)\,|\,\nu(p,t)\rangle\\ &= \mathrm{H}(p,t)\\ &= \langle \nu(p,t)\,|\,\nu_0(p)\rangle \Delta_{g(t)} f(p,t) + P(p, f(p,t), \nabla f(p,t))\,,\end{aligned}$$

thus, if we divide both sides by $\langle \nu(p,t)\,|\,\nu_0(p)\rangle$, which we can assume to be nonzero for a small positive time, we get

$$\begin{aligned}\frac{\partial f(p,t)}{\partial t} &= \Delta_{g(t)} f(p,t) + \frac{P(p, f(p,t), \nabla f(p,t))}{\langle \nu(p,t)\,|\,\nu_0(p)\rangle}\\ &= \Delta_{g(t)} f(p,t) + Q(p, f(p,t), \nabla f(p,t))\end{aligned}$$

where $Q(p,\,\cdot\,,\,\cdot\,)$ is a smooth function when its arguments are small. Moreover, as the coefficients of $\Delta_{g(t)}$ smoothly converge to the coefficients of $\Delta_{g(0)}$ as $t \to 0$, for t small the operators $\Delta_{g(t)}$ are uniformly strictly elliptic.
Then, the smooth function $f : M \times [0,T) \to (-\varepsilon, \varepsilon)$ solves the following partial differential *equation* (before we had to deal with a *system* of PDE's)

$$\begin{cases}\frac{\partial f}{\partial t}(p,t) = \Delta_{g(t)} f(p,t) + Q(p, f(p,t), \nabla f(p,t))\\ f(p,0) = 0\end{cases} \tag{1.5.2}$$

if and only if $\varphi(p,t) = \varphi_0(p) + f(p,t)\nu_0(p)$ is a solution of system (1.5.1) for the initial compact embedding φ_0, for $t \in [0,T)$.
This PDE is a quasilinear strictly parabolic equation, by what we said about the uniform ellipticity of $\Delta_{g(t)}$, in particular it is not degenerate (in some sense, passing to the height function f we "killed the degeneracy" of systems (1.3.1) and (1.5.1)) hence, we can apply the (almost standard) theory of quasilinear parabolic PDE's to have a smooth solution in a positive interval of time, continuously depending in C^∞ on the coefficients of the operator (which smoothly depend on the initial data φ_0).
The proof of a general theorem about existence, uniqueness and continuous dependence of a solution for a class of problems including (1.5.2) can be found again in [73] (see Appendix A).
Using the unique solution f of problem (1.5.2) we consider the associated map $\varphi = \varphi_0 + f\nu_0$, we possibly restrict the time interval in order that φ_t are all immersions with image in Ω and we apply Corollary 1.3.5 to reparametrize globally the hypersurfaces in a unique way in order to get a smooth solution of system (1.3.1) in some time interval.

Conversely, if we have a solution φ of system (1.3.1), for a small time interval $[0,T)$ the hypersurfaces φ_t are all embedded in the tubular neighborhood Ω of $\varphi_0(M)$ and representable as graph of a uniquely defined height function f, as above. Indeed, if we consider the smooth function $L : M \times [0,T) \to M$ given by

$$L(p,t) = \pi_M[\Psi^{-1}(\varphi(p,t))]\,,$$

where π_M is the projection map on the first factor of $M \times (-\varepsilon, \varepsilon)$, it is not difficult to see that all the maps $L_t : M \to M$ given by $L_t(p) = L(p,t)$ are diffeomorphisms, hence we define a function f as

$$f(q,t) = \pi_{(-\varepsilon,\varepsilon)}[\Psi^{-1}(\varphi(L_t^{-1}(q),t))]$$

for every $q \in M$ and $t \in [0,T)$. Then, the map $\widetilde{\varphi} = \varphi_0 + f\nu_0$ satisfies

$$\begin{aligned}\widetilde{\varphi}(L_t(p),t) &= \varphi_0(L_t(p),t) + f(L_t(p),t)\nu_0(L_t(p),t)\\ &= \varphi_0(\pi_M[\Psi^{-1}(\varphi(p,t))],t) + \pi_{(-\varepsilon,\varepsilon)}[\Psi^{-1}(\varphi(p,t))]\nu_0(\pi_M[\Psi^{-1}(\varphi(p,t))],t)\\ &= \varphi(p,t)\,,\end{aligned}$$

that is, $\widetilde{\varphi}$ and φ are obtained each other by composition with a time–dependent family of diffeomorphisms, in particular, they represent the same hypersurface in $\mathbb{R}^{n+1}$ at every time. Moreover, by the second part of Proposition 1.3.4, as φ is a solution of system (1.3.1), then the map $\widetilde{\varphi}$ is a solution of system (1.5.1), hence, by the above computations, f must solve system (1.5.2).

If we have two solutions φ and ψ of system (1.3.1), then, the two respectively associated height functions (via the above argument) must coincide, by the uniqueness of the solution of system (1.5.2). Hence, also the two associated maps $\widetilde{\varphi}$ and $\widetilde{\psi}$ coincide and they are given by compositions of φ and ψ with some time–dependent family of diffeomorphisms. This clearly implies that also φ and ψ can be obtained each other by composition with a time–dependent family of diffeomorphisms, that is $\psi(p,t) = \varphi(\Phi_t(p),t)$, with clearly $\Phi_0 = \mathrm{Id}_M$. By the argument in the proof of the second part of Proposition 1.3.4, since both φ and ψ are solutions of system (1.3.1) (their motion is only in normal direction), all the diffeomorphisms $\Phi_t : M \to M$ are the identity Id_M, hence $\varphi = \psi$. This proves the uniqueness of the solution of system (1.3.1).

If the initial hypersurface φ_0 is not embedded, that is, it has self–intersections, since locally every immersion is an embedding, we only need a little bit more care in the definition of the height function associated to a mean curvature flow (a regular tubular neighborhood is missing), in order to define the correspondence between a map φ and the associated height function f, then the same argument gives the conclusion also in the nonembedded case. □

Remark 1.5.4. This theorem gives the existence and uniqueness of the mean curvature flow in the case of a *compact* initial hypersurface. The noncompact case is more involved, as one needs estimates on the initial hypersurface (like similarly, on the initial datum in order to deal with the heat equation in all $\mathbb{R}^n$) to have existence in some positive interval of time. One possibility is to assume a uniform control on the norm of the second fundamental form of the initial hypersurface (see [39]).
Actually, by means of interior estimates (see Appendix B) Ecker and Huisken in [39] showed that a *uniform local Lipschitz condition* on a hypersurface is sufficient to guarantee short time existence. Another remarkable consequence of their work is the fact that the entire graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ has a *unique* smooth global mean curvature evolution for every time, remaining always a graph, see [38, 39] (notice that the same statement is not true for the heat equation without assuming some growth condition at infinity).
Similar interior estimates, depending only on a local bound on the value of a *function* (not on its gradient) whose graph is moving by mean curvature, were obtained by Colding and Minicozzi [29].

The uniqueness of the evolution by mean curvature of a noncompact initial hypersurface is another delicate point, like for the heat equation in $\mathbb{R}^n$. One possibility is to restrict the class of "admissible" evolutions, in order to have uniqueness, to the ones with a uniform bound (local in time) on the second fundamental form along the flow, see anyway [23] for the strongest result in this context.

Remark 1.5.5. The apparent loss of uniqueness of the flow if one consider the evolution by mean curvature of a hypersurface given as a subset of $\mathbb{R}^{n+1}$ (see Remark 1.3.6), due to the arbitrariness in choosing the parametrization, can actually be dealt with by noticing that the system (1.3.1) is invariant by reparametrization. Hence, even if possibly the immersions describing the hypersurfaces at time t are different, the hypersurfaces in $\mathbb{R}^{n+1}$ are however the same, that is, the flow is "geometrically" unique.

1.6 Other Second Order Flows

Let $\mathrm{S} = \mathrm{S}(\lambda_1, \dots, \lambda_n)$ be a symmetric function of the principal curvatures. Given an initial smooth immersion $\varphi_0 : M \to \mathbb{R}^{n+1}$ of the n–dimensional manifold M, one can consider the more general evolution problem

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{S}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{1.6.1}$$

where $\mathrm{S}(p,t)$ is a short way to denote the value of S associated to the curvatures $\lambda_1,\dots,\lambda_n$ of the hypersurface φ_t at the point $p \in M$. Besides the mean curvature flow, which is given by the choice $\mathrm{S} = \lambda_1 + \cdots + \lambda_n$, other studied cases are the *Gauss curvature flow*, where $\mathrm{S} = \lambda_1\lambda_2\cdots\lambda_n = \det \mathrm{A}$ is the Gauss curvature and the *inverse mean curvature flow* considering $\mathrm{S} = (\lambda_1+\cdots+\lambda_n)^{-1} = 1/\mathrm{H}$ (see [72], for instance).

For all these flows, we have the following existence result [73], which follows along the same line of Theorem 1.5.1.

Theorem 1.6.1. *Let M be compact and assume that at every point $p \in M$ we have*

$$\frac{\partial \mathrm{S}}{\partial \lambda_i}(\lambda_1(p),\dots,\lambda_n(p)) > 0, \qquad i = 1,\dots,n, \tag{1.6.2}$$

for an initial hypersurface φ_0, then system (1.6.1) *has a unique smooth solution in some positive time interval.*

It can be checked that condition (1.6.2) is equivalent to the parabolicity of the PDE's system at the initial time. In the case of the mean curvature flow such condition is satisfied for every initial hypersurface. For other flows one possibly needs to restrict the initial hypersurfaces to certain classes. For instance, the above result ensures the well–posedness of the Gauss curvature flow only when all the eigenvalues λ_i are positive everywhere on the initial hypersurface, that is, when it is strictly convex.