Appendix C

Hamilton's Maximum Principle for Tensors

Let V a vector bundle over a compact manifold M. Let h be a fixed metric on V, g a Riemannian metric on M and L a connection on V compatible with h. Both g and $L = \{L_{i\alpha}^{\beta}\}$ may depend on time t. We can form the Laplacian of a section f of V as the trace of the second covariant derivative with respect to g, using the connection L on V and the Levi–Civita connection on TM.

Let U an open subset of V and $\Psi(f)$ a vector field on V tangent to the fibers. We consider the nonlinear PDE

$$\partial_t f = \Delta f + \Psi(f) \tag{PDE}$$

and the ODE

$$\partial_t f = \Psi(f)\,. \tag{ODE}$$

Theorem C.1.1 (Hamilton [57, Section 4]). *Let X be a closed subset of $U \subset V$ invariant under parallel transport by the connection L and such that every fiber of X is convex.*
If every solution of the ODE starting in a fiber of X remains in X, then also any solution of the PDE remains in X.

Theorem C.1.2 (Hamilton [57, Section 8]). *Let f be a smooth section of V satisfying $\partial_t f = \Delta f + \Psi(f)$. Let $Z(f)$ be a convex function on the bundle, invariant under parallel transport whose level curves $Z(f) \leq \lambda$ are preserved by the ODE. Then, the inequality $Z(f) \leq \lambda$ is also preserved by the PDE for any constant λ.*
Moreover, if at time $t = 0$ at some point we have $Z(f) < \lambda$, then $Z(f) < \lambda$ everywhere on M at every time $t > 0$.

Theorem C.1.3 (Hamilton [57, Section 8]). *Let B be a symmetric bilinear form on V. Suppose that B satisfies the parabolic equation $\partial_t \mathrm{B} = \Delta \mathrm{B} + \Psi(\mathrm{B})$ where the matrix $\Psi(\mathrm{B}) \geq 0$ for all $\mathrm{B} \geq 0$.*
Then, if $\mathrm{B} \geq 0$ at time $t = 0$ it remains nonnegative definite for $t \geq 0$. Moreover, there exists an interval $0 < t < \delta$ on which the rank of B is constant and the null space of B is invariant under parallel transport and invariant in time, finally it also lies in the null space of $\Psi(\mathrm{B})$.

A good reference for these results is the book [28].